1.1 BASIC MODELS AND DIRECTION FIELDS

What is a Differential Equation and why do we study them?

Many things in life ranging from nuclear physics to love affairs involves changes in on quantity in relation to another. Since we understand rates of change as "differentials" it is natural to model these phenomena as differential equations. The book has many good examples, but lets look at an example that's not in the book.

Ex: Consider carbon dating. We know that all living things contain C_{12} and C_{14}, however when living things die the C_{14} starts to decay because it is radioactive. We can model this decay. First let t be the time that has elapsed since the death of the body. Also, let $x(t)$ be the amount of C_{14} left after a period of time t. We know the decay is linear, so our rate of change will be governed by the following equation,

$$\frac{dx}{dt} = -kx,$$

where k is the rate constant. Basically the C_{14} decay at a rate of kx [mass/time].

We also discussed direction fields for this problem, which is given bellow. For the field first calculate $dx/dt = 0$, called the "equilibrium solution" or "fixed point", which is the easiest solution to find. Then after plotting that slope for the respective x, t values find what the other slopes are for other relevant x and t values.
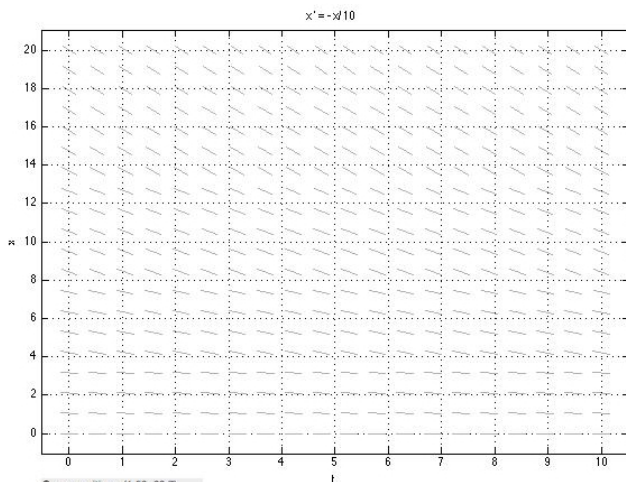
The next example is a very important example that appears on exams frequently.

21) a) We want to find the rate at which the amount of chemical is changing. When we develop models one should always keep track of the dimensions. It provides you with information and reduces the chance of making a mistake.

Now, let $x(t)$ be the amount of chemical in grams at a time t in hours. We know that the rate will be dx/dt, but to find the model we must realize that Total Rate = (Rate in) - (Rate out). Notice that (Rate in) = (.01 g/gal)x(300 gal/h) = 3 g/h, and (Rate out) = (300 gal/h)x(x/(1 Million) g/gal) = (3/1000)x g/h. So, our model becomes,

$$\frac{dx}{dt} = 3 - (3 \times 10^{-4})x.$$

b) What are they asking for this problem? They want to know if the amount of the chemical blows up or converges to something. To do this we look for an equilibrium solution,

$$\frac{dx}{dt} = 3 - (3 \times 10^{-4})x = 0 \Rightarrow x = 10^4.$$

For the next couple of problems they want us to find the ODE that gives us the behavior delineated in the problem,

7) $a \cdot 3 + b = 0 \Rightarrow b = -3a$, so in general the equation will be $y' = -ay + 3a$ since the solutions "approach".

9) $a \cdot 2 + b = 0 \Rightarrow b = -2a$, so in general the equation will be $y' = ay - 2a$ since the solutions "diverge from".

The next problem is to find the direction field.

12) We first find the equilibrium solutions of $y' = -y(5 - y)$, which are $y_* = 0$ and $y_* = 5$. To find the "stability" of the solutions, which means whether or not the solution diverges or converges to an equilibrium solution, we employ the first derivative test. This gives the following direction field,
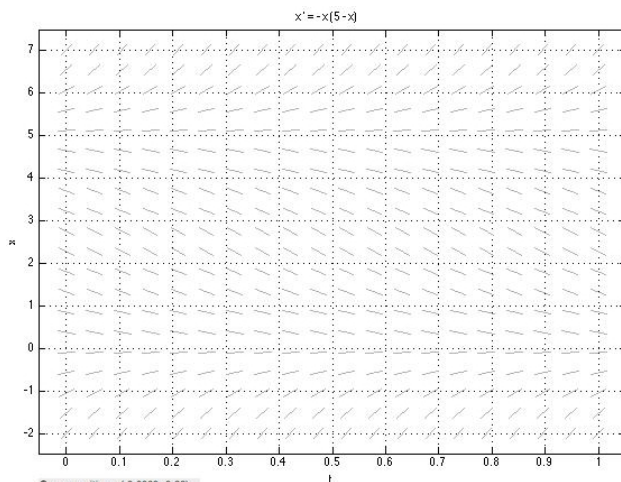
1.3 Classification of Differential Equations

We discussed the difference between ODEs and PDEs in class. In this course we only concentrate on ODEs.

Definition 1. The <u>order</u> of an ODE is the order of the highest derivative.

In the previous section we dealt with all first order ODEs. I gave some examples of higher order ODEs in class, and there are plenty of examples for that in the book.

Definition 2. Consider the ODE

$$F(t, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

then the ODE is said to be <u>linear</u> if F is a linear function with respect to $y, y', \ldots, y^{(n)}$.

Definition 3. We say an ODE is <u>nonlinear</u> if it is not linear.

Again we did examples of these in class and there are many examples in the book.

Definition 4. A function $y(t)$ is said to be a <u>solution</u> on (a, b) if for every $t \in (a, b)$, $y(t), y'(t), \ldots, y^{(n)}(t)$ exists and satisfies $F(t, y, y', \ldots, y^{(n)}) = 0$.

Notice, a solution need not be unique. We discussed examples of this in class.

For the next few examples we will verify a certain function is a solution to the given ODE. For these problems you want to first check the existence of the derivatives and then plug into the ODE to verify the ODE is satisfied.

7) a) $y_1' = y_1'' = e^t$ and $y_1'' - y_1 = e^t - e^t = 0$.
    b) $y_2' = \sinh t$ and $y_2'' = \cosh t$, furthermore $y_2'' - y_2 = \cosh t - \cosh t = 0$.

9) $y' = 3 + 2t$ and $ty' - y = (3t + 2t^2) - (3t + t^2) = t^2$.

10) a) $y_1' = 1/3$, $y_1'' = y_1''' = y_1^{(4)} = 0$ and $y^{(4)} + 4y''' + 3y = 3 \cdot t/3 = t$.
     b) $y_2' = -e^{-t} + 1/3$, $y_2'' = y^{(4)} = -y''' = e^{-t}$ and $y^{(4)} + 4y''' + 3y = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$.

Separable equations are the easiest equations to solve. This why it's extremely important to recognize separable equations. It will save you a lot of work! One thing we will notice right away is that Autonomous first order ODEs are always separable.

Definition 5. An ODE is <u>separable</u> if it can be written in the form $f(x)dx = g(y)dy$.

For the next few problems we will solve some separable equations.

1) We separate the equation by "moving" y to the left and dx to the right,

$$ydy = x^2dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{3}x^3 + C_0 \Rightarrow y = \pm\sqrt{\frac{2}{3}x^3 + C_1};\ y \neq 0.$$

4) We separate the equation by moving $3 + 2y$ to the left and dx to the right,

$$(3 + 2y)dy = (3x^2 - 1)dx \Rightarrow 3y + y^2 = x^3 - x + C;\ y \neq -\frac{3}{2}.$$

10) This type of problem is called an "Initial Value Problem" (IVP). The idea is to use the Initial Value to solve for the constant of integration. First we separate the problem by moving y to the left and dx to the right,

$$ydy = (1 - 2x)dx \Rightarrow \frac{1}{2}y^2 = x - x^2 + C.$$

Now, the initial value tells us that $y = -2$ when $x = 1$, so if we plug this into the above equation we get that $C = 2$, so plugging it back in and solving for y gives,

$$y = -\sqrt{2x - 2x^2 + 4};\ y \neq 0.$$

Notice we only chose the negative branch of the root because the initial condition starts with negative for the y value and we know that $y \neq 0$ so the solution can't magically cross into the positive branch, so we must stay on the negative branch for all time.

For part b and c, we did the plot in class, and the domain of existence is $-1 < x < 2$.

28 While we can try to do part a without solving, we can't do b without it.

Separating gives us,

$$\int \frac{dy}{y(4 - y)} = \int \frac{tdt}{1 + t} \Rightarrow \frac{1}{4}\int\left(\frac{1}{y} + \frac{1}{4 - y}\right)dy = \int \frac{u - 1}{u}du \Rightarrow \frac{1}{4}[\ln|y| - \ln|4 - y|] = u - \ln|u| + C_0$$

$$\Rightarrow \ln\left|\frac{y}{4 - y}\right| = 4t - 4\ln|1 + t| + C_1 \Rightarrow \frac{y}{4 - y} = e^{4t}\frac{K}{(1 + t)^4};$$

plugging in the initial condition gives us $K = y_0/(4 - y_0)$, then the full solution is

$$y = \frac{\frac{4e^{4t}y_0}{(1+t)^4(4-y_0)}}{1 + \frac{e^{4t}y_0}{(1+t)^4(4-y_0)}} \tag{2}$$

(a) As $t \to \infty$, $y \to 4$.

(b) $y_0 = 2 \Rightarrow K = 1$, so when $y = 3.99$, $T \approx 2.84367$ (via wolfram).

(c) Now, if $t = 2$, $y/(4 - y) = e^8 e^{-4\ln 3}K$, then $y = 3.99 \Rightarrow y_0 \approx 3.6622$ and $y = 4.1 \Rightarrow y_0 \approx 4.4042$. This gives us an interval of $3.6622 \leq y_0 \leq 4.4042$.