6.3 Step Functions (Discontinuous Forcing)

Recall the definition for a step function

$$u_c(t) = \begin{cases} 0, & \text{for } t < c, \\ 1, & \text{for } t \geq c; \end{cases} \tag{1}$$

Lets find the Laplace Transform

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{1}{s} e^{-cs}. \tag{2}$$

Now lets consider a forcing function where there is some variable forcing $f(t)$ after time $t = c$, then we have that the forcing is $f(t-c)u_c(t)$. Lets find the Laplace Transform of this

$$\begin{aligned} \mathcal{L}\{f(t-c)u_c(t)\} &= \int_0^\infty e^{-st} f(t-c) u_c(t) dt = \int_c^\infty e^{-st} f(t-c) dt = \int_0^\infty e^{-s(\tau+c)} f(\tau) d\tau \\ &= e^{-cs} \int_0^\infty e^{-s\tau} f(\tau) d\tau = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s). \end{aligned} \tag{3}$$

Problems 3 and 6 involve plotting and I showed how to plot them on the lecture. I'll put the other ones here

13) Here $f(t) = (t-2)^2 u_2(t)$ and $\mathcal{L}\{t^2\} = 2/s^3$, then $\mathcal{L}\{f(t)\} = 2e^{-2s}/s^3$.
18) Here $\mathcal{L}\{t\} = 1/s^2$, so $\mathcal{L}\{f(t)\} = 1/s^2 - e^{-s}/s^2$.
22) Here $c = 2$, so $G(s) = 2/(s^2 - 2^2)$, then $g(t) = \sinh 2t$, hence $f(t) = u_2(t)\sinh(2(t-2))$.

6.4 IVPs with Discontinuous Forcing

We discussed discontinuous forcing last time. Lets now do a bunch of problems

1) Here $f(t) = 1 - u_{3\pi}(t)$, then the Laplace Transform of the full IVP is

$$\begin{aligned} &-\cancelto{1}{y'(0)} - s\cancelto{0}{y(0)} + s^2 Y + Y = \frac{1}{s} - \frac{1}{s} e^{-3\pi s} \Rightarrow (s^2+1)Y = 1 + \frac{1}{s} + \frac{1}{s} e^{-3\pi s} \\ &\Rightarrow Y = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)} e^{-3\pi s} = \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} + \left(\frac{1}{s^2+1} + \frac{1}{s}\right) e^{-3\pi s} \\ &\Rightarrow y = \sin t + 1 - \cos t - [1 - \cos(t - 3\pi)] u_{3\pi}(t) = \sin t + 1 - \cos t - [1 + \cos t] u_{3\pi}(t). \end{aligned}$$

4) We take the Laplace Transform of the full IVP

$$\begin{aligned} -\cancelto{0}{y'(0)} - s\cancelto{0}{y(0)} + s^2 Y + Y &= \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-\pi s} \Rightarrow Y = \frac{1}{(s^2+4)(s^2+1)} + \frac{1}{(s^2+4)(s^2+1)} e^{-\pi s} \\ &\Rightarrow Y = \frac{1}{3} \cdot \frac{1}{s^2+1} - \frac{1}{6} \cdot \frac{2}{s^2+2^2} + \left(\frac{1}{3} \cdot \frac{1}{s^2+1} - \frac{1}{6} \cdot \frac{2}{s^2+2^2}\right) e^{-\pi s} \\ &\Rightarrow y = \frac{1}{3} \sin t - \frac{1}{6} \sin 2t + \left(\frac{1}{3} \sin(t - \pi) - \frac{1}{2} \sin(2t - 2\pi)\right) u_\pi(t). \end{aligned}$$

6) We take the Laplace Transform

$$\begin{aligned} &-\cancelto{1}{y'(0)} - s\cancelto{0}{y(0)} + s^2 Y - 3\cancelto{0}{y(0)} + 3sY + 2Y = \frac{1}{2} e^{-2s} \Rightarrow (s^2+3s+2)Y = 1 + \frac{1}{s} e^{-2s} \\ &\Rightarrow Y = \frac{1}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)} e^{-2s} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2} - \frac{2}{s+1}\right) e^{-2s} \\ &\Rightarrow y = e^{-t} - e^{-2t} + \left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2} e^{-2(t-2)}\right) u_2(t). \end{aligned}$$

8) Taking the Laplace Transform gives

$$-\cancelto{0}{y'(0)}-s\cancelto{0}{y(0)}+s^2Y-\cancelto{0}{y(0)}+sY+\frac{5}{4}Y=\frac{1}{s^2}-\frac{1}{s^2}e^{-\pi s/2}\Rightarrow (s^2+s+5/4)Y=\frac{1}{s^2}-\frac{1}{s^2}e^{-\pi s/2}$$

$$\Rightarrow Y=\frac{1}{s^2(s^2+s+5/4)}-\frac{1}{s^2(s^2+s+5/4)}e^{-\pi s/2}.$$

Now we do the partial fractions

$$\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+s+5/4}\Rightarrow As^3+As^2+\frac{5}{4}As+Bs^2+Bs+\frac{5}{4}B+Cs^3+Ds^2=(A+C)s^3+(A+B+D)s^2+(B+5A/4)s+\frac{5}{4}B=1.$$

Then we get $B=4/5$, $A=-16/25$, $C=16/25$, and $D=-4/25$, then

$$Y=\left[-\frac{16}{25}\cdot\frac{1}{s}+\frac{4}{5}\cdot\frac{1}{s^2}\;+\frac{16}{25}\cdot\frac{s-1/4}{s^2+s+5/4}\right]\left(1-e^{-\pi s/2}\right).$$

That final term in the brackets is going to take more effort

$$\frac{s-1/4}{s^2+s+5/4}=\frac{s-1/4}{s^2+s+1/4+1}=\frac{s+1/2}{(s+1/2)^2+1}-\frac{3/4}{(s+1/2)^2+1}.$$

Then the solution is

$$y=\frac{16}{25}\left(e^{-t/2}\cos t-\frac{3}{4}e^{-t/2}\sin t+\frac{5}{4}t-1\right)-\frac{16}{25}u_{\pi/2}(t)\left(e^{-(t-\pi/2)/2}\cos(t-\pi/2)-\frac{3}{4}e^{-(t-\pi/2)/2}\sin(t-\pi/2)\;+\frac{5}{4}(t-\pi/2)-1\right).$$

6.5 Impulse Functions

An impulse is a change of momentum over a period of time, such as hitting a baseball. The momentum

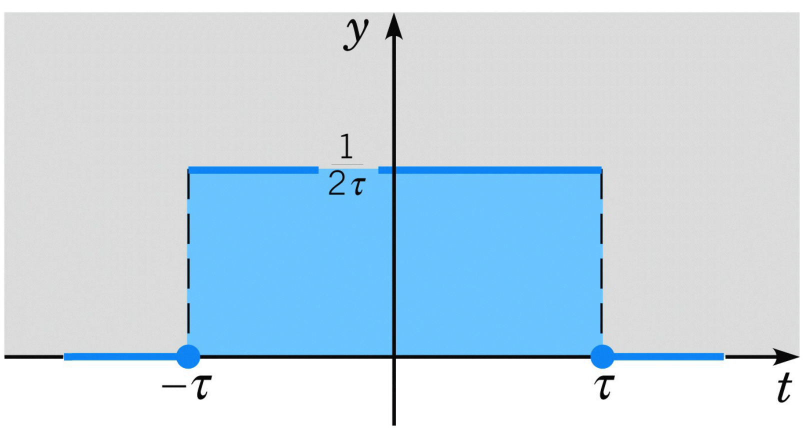


Figure 6.5.1

here is (I prefer using ϵ instead of τ)

$$p=\int_{-\infty}^{\infty}g(t)dt=\int_{-\epsilon}^{\epsilon}\frac{1}{2\epsilon}dt=1.$$

Notice that we can make τ smaller and keep the momentum at $p=1$ such as in the following plot In fact,

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}g(t)dt=1.$$

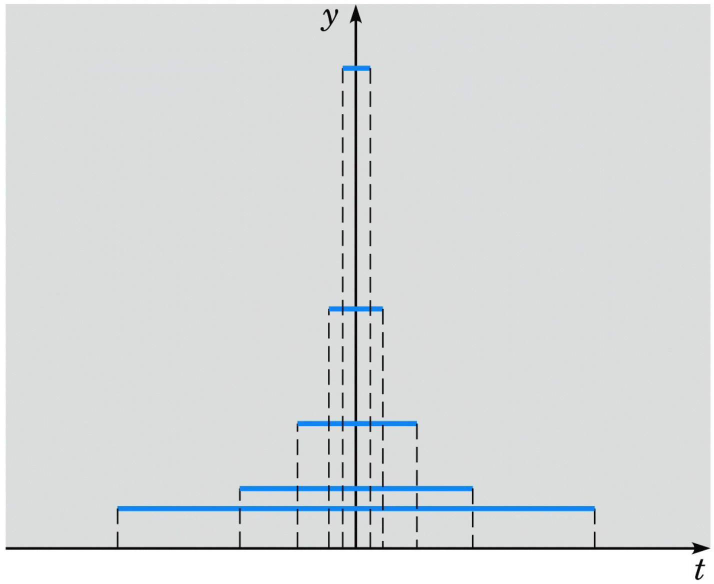


Figure 6.5.2

Notice this is 0 everywhere except at $t = 0$. Now if we can do this at $t = 0$, we can define a "function" with this porperty for any $t = t_0$,

$$\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1; \;\delta(t - t_0) = 0 \forall t \neq t_0 \tag{4}$$

called the <u>Dirac delta function</u>, however this isn't a function, but rather a distribution. Doing this for $t_0 > 0$ will allow us to employ Laplace Transforms. Notice that we can write the delta function as the following limit,

$$\delta(t - t_0) = \lim_{\epsilon \to 0} \begin{cases} 0 & t \leq t_0 - \epsilon, \\ \frac{1}{2\epsilon} & t_0 - \epsilon < t < t_0 + \epsilon, \\ 0 & t \geq t_0 + \epsilon; \end{cases} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}(u_{t_0-\epsilon}(t) - u_{t_0+\epsilon}(t)).$$

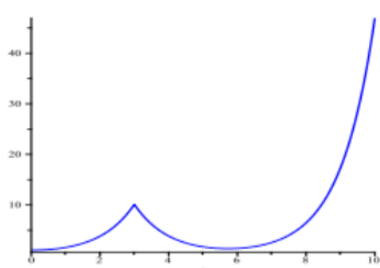
Now we take the Laplace

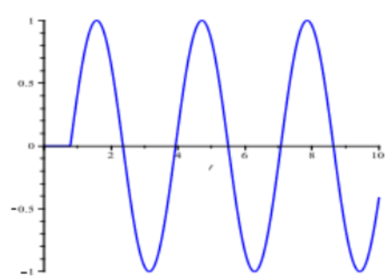
$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \cdot \frac{1}{s}\left(e^{(-t_0+\epsilon)s} - e^{(-t_0-\epsilon)s}\right) = e^{-t_0 s} \lim_{\epsilon \to 0} \frac{1}{\epsilon s} \cdot \frac{1}{2}\left(e^{\epsilon s} - e^{-\epsilon s}\right) = e^{-t_0 s} \cancelto{1}{\lim_{\epsilon \to 0} \frac{\sinh \epsilon s}{\epsilon s}} = e^{-t_0 s}. \tag{5}$$

Now lets do some problems

4) We solve the IVP and plot it (on the left)

$$-\cancelto{0}{y'(0)} - \cancelto{1}{sy(0)} + s^2 Y - Y = -20e^{-3s} \Rightarrow (s^2 - 2)Y = -20e^{-3s}$$
$$\Rightarrow Y = \frac{1}{s^2 - 1}\left(-20e^{-3s} + s\right) \Rightarrow y = \cosh t - 20 \sinh(t - 3)u_3(t).$$





8) We solve the IVP and plot it (on the right)

$$
\begin{aligned}
&-\cancelto{0}{y'(0)} - s\cancelto{0}{y(0)} + s^2Y + 4Y = 2e^{-(\pi/4)s} \Rightarrow Y = \frac{2}{s^2+4}e^{-(\pi/4)s} \\
&\Rightarrow y = \sin(2(t-\pi/4))u_{\pi/4}(t) = u_{\pi/4}(t)\cos 2t.
\end{aligned}
$$

11) As per usual,

$$
(s^2+2s+2)Y = \frac{s}{s^2+1} + e^{-(\pi/2)s} \Rightarrow Y = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-(\pi/2)s}}{s^2+2s+2}.
$$

We employ partial fractions,

$$
\begin{aligned}
&\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2} \\
&\Rightarrow As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Cs + Ds^2 + D = s \\
&\Rightarrow (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D) = s.
\end{aligned}
$$

From this we get $A = 1/5 = -C$, $B = 2/5$, and $D = -4/5$, so

$$
Y = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right] + e^{-(\pi/2)s}\frac{1}{(s+1)^2+1}.
$$

Furthermore,

$$
\frac{s+4}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}.
$$

Then,

$$
y = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}(\cos t + 3\sin t) + e^{-(t-\pi/2)}\sin(t-\pi/2)u_{\pi/2}(t).
$$