2.1 Linear Equations; Method of Integration Factor

Consider the ODE $\frac{dy}{dx} - \frac{y}{x} + \frac{f(x)}{x} = 0$. This is clearly not separable.

Now consider the ODE $t^2\frac{dx}{dt} + 2xt = t$. This too is not separable, but we can make it separable by employing a small trick. Notice that $t^2\frac{dx}{dt} + 2xt = \frac{d}{dt}(xt^2)$, so the ODE becomes, $\frac{d}{dt}(xt^2) = t$, which is separable. This is what is referred to as an "exact ODE". So we get,

$$\frac{d}{dt}(xt^2) = t \Rightarrow d(xt^2) = tdt \Rightarrow \int d(xt^2) = \int tdt \Rightarrow xt^2 = \frac{1}{2}t^2 + C \Rightarrow x = \frac{1}{2} + Ct^{-2}.$$

This is the idea. If we encounter an equation that isn't separable we need to change it in some way that makes it separable.

Lets look at the first equation again and write it in differential form, i.e.

$$\frac{dy}{dx} - \frac{y}{x} + \frac{f(x)}{x} = 0 \Rightarrow xdy - ydx + f(x)dx = 0.$$

Notice, that $xdy - ydx$ is almost quotient rule, we just need to finish the denominator, which we notice should be x^2, so let's multiply through by $1/x^2$,

$$\frac{xdy - ydx}{x^2} + \frac{f(x)}{x^2}dx = 0 \Rightarrow d\left(\frac{y}{x}\right) = -\frac{f(x)}{x^2}dx \Rightarrow \int d\left(\frac{y}{x}\right) = -\int \frac{f(x)}{x^2}dx$$
$$\Rightarrow \frac{y}{x} = -\int \frac{f(x)}{x^2}dx \Rightarrow y = -x\int \frac{f(x)}{x^2}dx.$$

This is called the method of "integrating factors", where $1/x^2$ is called the "integrating factor", which are delineated in the following definition.

Definition 1. Consider an ODE of the form

$$\frac{dy}{dx} + p(x)y = g(x). \tag{1}$$

We call $\mu(x)$ an <u>integrating factor</u> if

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right]$$

is an exact ODE, i.e.

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right] \Leftrightarrow d(\mu(x)y) = \mu(x)g(x)dx. \tag{2}$$

All we need to do now is figure out what $\mu(x)$ is in general, but fortunately Leibniz already did that for us,

$$\mu(x) = \exp\left(\int^x p(\xi)d\xi\right). \tag{3}$$

In the following examples we use the method of integrating factors to solve our ODE,

5c) The integrating factor is $\mu = \exp\left(\int^t -2ds\right) = e^{-2t}$. Now, we use our method to get,

$$e^{-2t}y = 3\int e^{-t}dt = -3\int e^{-t}dt = -3e^{-t} + C \Rightarrow y = -3e^t + Ce^{2t}.$$

Now, notice if $C > 0$, $y \to \infty$ as $t \to \infty$, and if $C \leq 0$, $y \to -\infty$ as $t \to \infty$.

10c) The integrating factor is $\mu = \exp\left(\int^t -1/sds\right) = 1/t$, then

$$\frac{y}{t} = \int e^{-t}dt = -e^{-t} + C \Rightarrow y = te^{-t} + Ct.$$

Now, notice if $C = 0$, $y \to 0$ as $t \to \infty$, and if $C \neq 0$, $y \to \infty$ as $t \to \infty$.

21b) The integrating factor is $\mu = \exp\left(\int^t -(1/2)ds\right) = e^{-t/2}$. Then,

$$e^{-t/2}y = 2\int e^{-t/2}\cos t dt = \frac{4}{5}e^{-t/2}(2\sin t - \cos t) + C.$$

We did the integration in class. Know how to do the integration! Then, we get

$$y = \frac{4}{5}(2\sin t - \cos t) + Ce^{t/2}.$$

From the initial condition we get $C = a + 4/5$. We see that the behavior of the system changes at $C = 0$, so $a_0 = -4/5$. Now, when $a = -4/5$, y is oscillatory as $t \to 0$, specifically $y \to \frac{4}{5}(2\sin t - \cos t)$. Furthermore, if $a < -4/5$, $y \to -\infty$, and if $a > -4/5$, $y \to \infty$.

30) The integrating factor is $\mu = \exp\left(-\int^t ds\right) = e^{-t}$. Then we get

$$e^{-t}y = \int\left(e^{-t} + 3e^{-t}\sin t\right)dt = -e^{-t} + 3\int e^{-t}\sin t dt.$$

We employ integration by parts for $\int e^{-t}\sin t dt$ with $u = e^{-t} \Rightarrow du = -e^{-t}dt$ and $dv = \sin t dt \Rightarrow v = -\cos t$,

$$\int e^{-t}\sin t dt = -e^{-t}\cos t + \int e^{-t}\cos t dt$$

We employ by parts again with $u = e^{-t} \Rightarrow du = -e^{-t}dt$ and $dv = \cos t dt \Rightarrow v = \sin t$,

$$\int e^{-t}\sin t dt = -e^{-t}\cos t + e^{-t}\sin t - \int e^{-t}\sin t dt \Rightarrow \int e^{-t}\sin t dt = -\frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t.$$

Plugging this back into our ODE gives,

$$e^{-t}y = -e^{-t} - \frac{3}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t + C \Rightarrow y = -1 - \frac{3}{2}\cos t + \frac{3}{2}\sin t + Ce^{t}.$$

The initial condition gives us $y_0 = -1 - 3/2 + C = -5/2 + C$, then our solution is

$$\Rightarrow y = -1 - \frac{3}{2}\cos t + \frac{3}{2}\sin t + (y_0 + 5/2)e^{t}. \tag{4}$$

Then for $y > -5/2$, $y \to \infty$ as $t \to \infty$. For $y < -5/2$, $y \to -\infty$ as $t \to \infty$. However, for $y_0 = -5/2$, y oscillates, but remains finite.

2.3 More Modeling Problems

There hasn't been any EE problems yet in the book, so lets do one,

Ex: Consider a Resister-Inductor (RL) circuit in series. Let x be the current at time t. Let V be the voltage across the voltage source, R be the resistance of the resistor, and L be the inductance of the inductor. Now, the voltage drop through the resistor is: $V_R = Rx$, and the voltage through the inductor is $V_L = Ldx/dt$. Now, by Kirchoff's law, we know that the voltages in a loop sum up, so $Ldx/dt + Rx = V$, and in standard form this is,

$$\frac{dx}{dt} + \frac{R}{L}x = \frac{V}{L}.$$

We can solve this via separation,

$$\int \frac{dx}{-Rx/L + V/L} = \int dt \Rightarrow -\frac{L}{R}\ln\left(\frac{V}{L} - \frac{R}{L}x\right) = t + C_0 \Rightarrow \ln\left(\frac{V}{L} - \frac{R}{L}x\right) = -\frac{R}{L}t + C_1$$
$$\Rightarrow \frac{V}{L} - \frac{R}{L}x = \exp\left(-\frac{R}{L}t + C_1\right) = e^{-Rt/L}e^{C_1} = k_0 e^{-Rt/L}$$
$$\Rightarrow \frac{R}{L}x = \frac{V}{L} - k_0 e^{-Rt/L} \Rightarrow x = \frac{V}{R} - k_1 e^{-Rt/L}.$$

Notice that we could use separation because V was constant, however if $V = V(t)$, then we would have to use integrating factor.

The next couple of examples are from the book,

3) Notice that there are two processes delineated in the problem. And the second process starts as soon as the first process ends. So we need to solve the first problem and then use information from the first problem to solve the second problem.

Process 1: Let x be the amount of salt in lb at time t min. The rate in will be (1/2) lb/gal $\times$ 2 gal/min = 1 lb/min. And the rate out is $x/200$ lb/gal $\times$ 2 gal/min $= x/50$ lb/min. Now notice that there is no salt in the tank when the process starts, so our full IVP becomes,

$$\frac{dx}{dt} = 1 - \frac{x}{50};\ x(0) = 0.$$

Now we solve this via separation,

$$\int \frac{dx}{1 - x/50} = \int dt \Rightarrow -50\ln(1 - x/50) = t + C_0 \Rightarrow \ln(1 - x/50) = -t/50 + C_1$$
$$\Rightarrow 1 - \frac{x}{50} = k_0 e^{-t/50} \Rightarrow x = 50 - k_1 e^{-t/50}.$$

Now, we solve for the constant from the initial condition,

$$x(0) = 50 - k_1 - 0 \Rightarrow k_1 = 50 \Rightarrow x = 50\left(1 - e^{-t/50}\right).$$

Now, since the process is stopped at $t = 10$ min. we need to calculate the amount at that time, $x(10) = 50\left(1 - e^{-1/5}\right)$.

Process 2: Now, in order to distinguish this process from the previous one, let y be the amount of salt in lb at time t min. Notice, no more salt is entering, so the rate in is zero. The rate out will be the same as before $y/50$ lb/min. For our initial conditions, notice that where this process begins the other one had ended, so $y(0) = x(10)$.

$$\frac{dy}{dt} = -\frac{y}{50};\ y(0) = 50\left(1 - e^{-1/5}\right).$$

Again, we solve this via separation,

$$\ln y = -\frac{1}{50}t + C \Rightarrow y = ke^{-t/50}.$$

From the initial condition we have,

$$y(0) = k = 50\left(1 - e^{-1/5}\right) \Rightarrow y = 50\left(1 - e^{-1/5}\right)e^{-t/50}.$$

Finally, the process stops after another 10 minutes, so $y(10) = 50\left(1 - e^{-1/5}\right)e^{-1/5}$.

8) For this problem we first realize that every year the bank statement increases by k \$ from what the person deposits. However, there is also an interest being earned, which is on the total amount. So, every year the increase due to interest is rS \$. This means the total rate is going to be, $dS/dt = k + rS$. And the initial condition will be $S(0) = 0$.

(a) We solve this via separation,

$$\int \frac{dS}{k + rS} = \int dt \Rightarrow \frac{1}{r}\ln(k + rS) = t + C_0 \Rightarrow \ln(k + rS) = rt + C_1 \Rightarrow k + rS = C_2 e^{rt}.$$

From the initial condition we get,

$$S(0) = 0 \Rightarrow C_2 = k \Rightarrow S = \frac{k}{r}\left(e^{rt} - 1\right).$$

(b) For this problem we solve for $S(40) = 10^6$ with $r = .075$. Plugging all these into the equation gives $k = (.075 \times 10^6)/[\exp(.075 \times 40) - 1]$.

(c) Plug in the values they give and then ask wolfram alpha to solve it.

Ex: Consider two connected 100 gal tanks. Tank 1 initially has 0 lb of salt and Tank 2 has 1 lb of salt. We start pumping 1/2 lb/gal of salt into Tank 1 at a rate of 2 gal/min. The mixture leaves Tank 1 and enters Tank 2 then finally leaves Tank 2 all at the same rate. Find the amount of salt for any time in Tank 1 and Tank 2.

Solution: Whenever you get something like this, it's best to separate the processes and deal with them individually. Let the amount of salt in Tank 1 be x and in Tank 2 be y. Notice that the rate of the amount of salt entering Tank 2 must be equivalent to the rate of salt leaving Tank 1. Now, we can make a table to visualize this.

	Rate in	Rate out	ODE
Tank 1	1 lb/min	$x/50$ lb/min	$dx/dt = 1 - x/50;\ x(0) = 0$
Tank 2	$x/50$ lb/min	$y/50$ lb/min	$dx/dt = x/50 - y/50;\ y(0) = 1$

Then we solve the ODEs

Tank 1:

$$50\int \frac{dx}{50 - x} = \int dt \Rightarrow -50\ln|50-x| = t + C_0 \Rightarrow \ln|50-x| = \frac{-t}{50} + C_1 \Rightarrow 50 - x = ke^{-t/50} \Rightarrow x = 50 - ke^{-t/50}.$$

Plugging in our initial condition gives, $x(0) = 50 - k = 0 \Rightarrow k = 50$. Then our solution is,

$$x = 50\left(1 - e^{-t/50}\right)$$

Tank 2:

$$\frac{dy}{dt} + \frac{y}{50} = 1 - e^{-t/50}.$$

The integrating factor is $\mu = \exp\left(\int^t ds/50\right) = e^{t/50}$. Plugging this into our integrating factor formula gives,

$$\int d(e^{t/50}y) = \int (e^{t/50} - 1)dt \Rightarrow e^{t/50}y = 50e^{t/50} - t + C \Rightarrow y = 50 - te^{-t/50} + Ce^{-t/50}.$$

Plugging in our initial condition gives us $y(0) = 50 + C = 1 \Rightarrow C = -49$. Then our solution is,

$$y = 50 - te^{-t/50} - 49e^{-t/50}.$$