3.1 Homogeneous Equations with Constant Coefficients

It should be noted that while this chapter is on second order ODEs, we will develop the theory for higher order ODEs because the theory is exactly the same! Let us first go over some definitions we might not know,

Definition 1. An ODE is homogeneous if it is of the form

$$p_n(t)y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_2(t)y''(t) + p_1(t)y'(t) + p_0(t)y(t) = 0. \tag{1}$$

So an example of a second order homogeneous ODE would be
$p_2y'' + p_1y' + p_0y = 0$.

Definition 2. An ODE is said to be nonhomogeneous if it's not homogeneous.

An example of a second order nonhomogeneous ODE would be $p_2y'' + p_1y' + p_0y = f(t)$. In this section we will only deal with constant coefficients which mean each $p_n(t) = a_n$ where $a_0, a_1, \ldots, a_{n-1}, a_n$ are all constants.

Now, we consider a special case of Eq. (1): $y' + ay = 0$ We know how to solve this, we simply use separation to get $y = ke^{-ax}$. So, we can "guess" that the form of the solutions for Eq. (1) with constant coefficients will be $y = ke^{rx}$. Now, we plug this guess in to see what the solutions exactly are. Notice that the nth derivative is, $y^{(n)} = kr^ne^{rx}$, so plugging this into (1) with $p_n(t) = a_n$ gives,

$$a_nkr^ne^{rx} + a_{n-1}kr^{n-1}e^{rx} + \cdots + a_2kr^2e^{rx} + a_1kre^{rx} + a_0ke^{rx} = 0$$
$$\Rightarrow ke^{rx}\left(a_nr^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0\right) = 0.$$

Now, all we have to do is solve the polynomial equation. Since this is an nth order polynomial, there will be n solutions, i.e. $r = r_1, r_2, \ldots, r_{n-1}, r_n$. Since the polynomial equation has n solutions, the ODE will also have n solutions, so by superposition we get,

$$y = k_1e^{r_1x} + k_2e^{r_2x} + \cdots + k_{n-1}e^{r_{n-1}x} + k_ne^{r_nx}.$$

We have just proved a theorem,

Theorem 1. *Consider the ODE*

$$a_ny^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_2y''(x) + a_1y'(x) + a_0y(x) = 0. \tag{2}$$

such that $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants. Then,

$$y = k_1e^{r_1x} + k_2e^{r_2x} + \cdots + k_{n-1}e^{r_{n-1}x} + k_ne^{r_nx}, \tag{3}$$

where $k_1, k_2, \ldots, k_{n-1}, k_n$ are constants and $r_1, r_2, \ldots, r_{n-1}, r_n$ satisfy the polynomial equation

$$a_nr^n + a_{n-1}r^{n-1} + \cdots + a_2r^2 + a_1r + a_0 = 0, \tag{4}$$

only if $r_1 \neq r_2 \neq \cdots \neq r_{n-1} \neq r_n$.

Definition 3. We call Eq. (4) the characteristic equation of ODE (2), and the polynomial is called the characteristic polynomial.

Now, lets do a few problems from the book,

1) The characteristic polynomial is $r^2 + 2r - 3$, so

$$r^2 + 2r - 3 = 0 \Rightarrow (r + 3)(r - 1) = 0 \Rightarrow r = 1, -3 \Rightarrow y = c_2e^x + c_2e^{-3x}.$$

7) The characteristic polynomial is $r^2 - 9r + 9$, so

$$r = \frac{1}{2}(9 \pm 3\sqrt{5}) \Rightarrow y = c_1e^{\frac{1}{2}(9+3\sqrt{5})x} + c_2e^{\frac{1}{2}(9-3\sqrt{5})x}.$$

12) The characteristic polynomial is $r^2 + 3r$, so

$$r = 0, -3 \Rightarrow y = c_1 + c_2e^{-3x},$$

and from the initial conditions we get $y = -1 - e^{-3x}$.

18) Here they give us the solution and we have to extract the ODE. Notice that from the solution we deduce

$$r = -\frac{1}{2}, -2 \Rightarrow (r + \frac{1}{2})(r + 2) = r^2 + \frac{5}{2}r + 1 = 0 \Rightarrow y'' + \frac{5}{2}y' + y = 0.$$

21) This is kind of a silly question, but since there is a similar one on the homework lets do it. We solve the ODE as per usual,

$$r^2 - r - 2 = (r - 2)(r + 1) = 0 \Rightarrow r = -1, 2 \Rightarrow y = c_1 e^{-x} + c_2 e^{2x}.$$

From the initial condition we have the equations $c_1 + c_2 = \alpha$ and $2c_2 - c_1 = 2$, so $3c_2 = \alpha + 2$. This means that if $\alpha = -2$, as $t \to \infty$, $y \to 0$. However, for the second part of the problem there are no solutions that always blow up because we have a negative exponential term that will persist.

24) For this problem the ODE itself has the parameter α. This leads to interesting conclusions without even solving, but the easiest most intuitive way to come to those conclusions will be by solving, even though it is more tedious and time consuming. We solve the ODE,

$$r^2 + (3 - \alpha)r - 2(\alpha - 1) = 0 \Rightarrow (r - (\alpha - 1))(r + 2) = 0 \Rightarrow r = -2, \alpha - 1 \Rightarrow y = c_2 e^{-2x} + c_2 e^{(\alpha - 1)x}.$$

So, for $\alpha < 1$, $y \to \infty$. If $\alpha = 1$, $y \to c_2$, and if $\alpha > 1$, and $y \to \pm\infty$ only if $c_2 \neq 0$.

2.7 Euler's Method

Numerical solutions to ODEs are all about approximating a derivative and using that to approximate the solution. What is the definition of the derivative and how do we approximate it? Think back to Calc I, we derived the definition of the derivative by using a slope and watching what happens when $\Delta t \to 0$. Lets use the formula for slope again for *first order* ordinary differential equations,

$$y'(t) = f(t, y) \Rightarrow f(t, y) \approx \frac{\Delta y}{\Delta t} = \frac{y - y_0}{t - t_0}.$$

Now lets evaluate f at t_1, y_1, then we get,

$$f(t_1, y_1) \approx \frac{y_1 - y_0}{t_1 - t_0} \Rightarrow y_1 - y_0 \approx (t_1 - t_0) f(t_0, y_0) \Rightarrow y_1 \approx y_0 + (t_1 - t_0) f(t_0, y_0).$$

Look at that! We just developed a formula to approximate y at t_1 by using the information we had for the system at t_0. If we can approximate the data at t_1 by using the previous time (i.e. t_0), why can't we do this for any time? That is we can approximate y at t_{n+1} via the formula, $y_{n+1} \approx y_n + \Delta t f(t_n, y_n)$. The standard way to write this however is with, $h = \Delta t$, basically a renaming and we usually use $y_0 = y(t_0)$, i.e. the initial condition, and we also drop the $\approx$ and us $=$. So our general formula is,

$$y_{n+1} = y_n + h f(t_n, y_n); \; y_0 = y(t_0). \tag{5}$$

When debugging your codes use the following example, and make sure your values are close to mine. Your values might be ever so slightly off, but not more than say .0001.

(1) $f(t, y) = 3 + t - y$, which gives us the equation $y_{n+1} = y_n + h \cdot (3 + t_n - y_n)$ where $y_0 = 1$.

(a) Here we have $h = 0.1$, so we have the following t's. We get them just by starting at t_0 and incrementing. $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.3$, $t_4 = 0.4$. Then we have, $y_1 = y_0 + h \cdot (3 + t_0 - y_0) = 1 + (0.1)(3 + 0 - 1) = 1.2$, $y_2 = y_1 + h \cdot (3 + t_1 - y_1) = 1.39$, $y_3 = y_2 + h \cdot (3 + t_2 - y_2) = 1.571$, and $y_4 = y_3 + h \cdot (3 + t_3 - y_3) = 1.7439$. Lets put this in a table to make it look pretty,

n	0	1	2	3	4
t_n	0	0.1	0.2	0.3	0.4
y_n	1	1.2	1.39	1.571	1.7439

(b) Hopefully part a gave you a good idea of how we do these problems, so I'll just give the table of values I received when running my code on matlab (remember $h = 0.05$):

n	0	1	2	3	4	5	6	7	8
t_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4
y_n	1	1.1	1.1975	1.2926	1.3855	1.4762	1.5649	1.6517	1.7366

(c) Here $h = 0.025$,

n	0	1	2	3	4	5	6	7	8
t	0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2
y	1	1.05	1.0994	1.1481	1.1963	1.2439	1.2909	1.3374	1.3833

n	9	10	11	12	13	14	15	16
t	0.225	0.25	0.275	0.3	0.325	0.35	0.375	0.4
y	1.4288	1.4737	1.5181	1.562	1.6055	1.6484	1.6910	1.7331

(d) Next we solve the equation via integrating factors to get $y = 2 + t - e^{-t}$, and calculating the points gives us the following comparison,

h	$t =$	0.1	0.2	0.3	0.4
0.1	$y(t) =$	1.2	1.39	1.571	1.7439
0.05	$y(t) =$	1.1975	1.3855	1.5649	1.7366
0.025	$y(t) =$	1.1963	1.3833	1.562	1.7331
Exact	$y(t) =$	1.19516	1.38127	1.55918	1.72968