3.5 Undetermined Coefficients

Consider the nonhomogeneous ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_2 y'' + a_1 y' + a_0 = f(x). \tag{1}$$

Notice that our usual solution won't work, but maybe it's part fo the solution. Suppose y_p is a solution to (1) that is linearly independent with respect to the solution the homogeneous ODE. Let y be the general solution of (1). Lets plug in $y_c = y - y_p$ in (1), then we get that y_c is a solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_2 y'' + a_1 y' + a_0 = 0. \tag{2}$$

So, in fact y_c is the solution to the homogeneous ODE, so $y = y_c + y_p$, where y_c is the homogeneous part of the solution and y_p is the purely nonhomogeneous part of the solution.

Definition 1. The <u>characteristic solution</u>, y_c, is the general solution of (2) and the <u>particular solution</u>, y_p, is the additional solution to (1).

Case1: No term in $f(x)$ is the same as any term in y_c. Then y_p is a linear combination of terms of $f(x)$ and their derivatives.

- $f_1(x) = x^n \Rightarrow y_{1_p} = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$. If our f is a polynomial, the particular solution will be of the form of the most general polynomial of order of the polynomial in f.
- $f_2(x) = e^{mx} \Rightarrow y_{2_p} = ke^{mx}$. This one is easy.
- $f_3(x) = \cos(mx)$ or $\sin(mx)$, then $y_{3_p} = A\cos(mx) + B\sin(mx)$. If we have sine or cosine our particular solution will be a linear combination of sines and cosines.
- $f(x) = f_1(x) + f_2(x) + f_3(x) \Rightarrow y_p = y_{1_p} + y_{2_p} + y_{3_p}$. If we have a combination of these simple examples then we just combine all of their respective particular solutions.
- $f(x) = f_1(x) f_2(x) f_3(x) \Rightarrow y_p = y_{1_p} y_{2_p} y_{3_p}$. We do the same sort of thing with products.

Case 2: $f(x)$ contains terms that are x^n times terms in y_c, i.e. if $u(x)$ is a term of y_c and $f(x)$ contains $x^n u(x)$. Then y_p is as usual but multiply by "x".

- Consider $y_c = g(x) + e^{mx}$ and $f(x) = l(x) + x^n e^{mx}$, where we don't care about $g(x)$ and $l(x)$ – we are just thinking of them as place holders. Then our particular solution is $y_p = h(x) + (A_n x^{n+1} + A_{n-1} x^n + \cdots + A_0 x) e^{mx}$.
- Consider a similar case except with sine, also equivalently cosine. $y_c = g(x) + \sin(mx)$ and $f(x) = l(x) x^n \sin(mx)$, then our particular solution is, $y_p = h(x) + (A_n x^{n+1} + A_{n-1} x^n + \cdots + A_0 x)(B\cos(mx) + C\sin(mx))$.

Case 3: If y_c contains repeated roots with the highest being of order λ (i.e. x^λ) and $f(x)$ contains terms x^n times the repeated root terms, then multiply out by $x^{\lambda+1}$.

- $y_c = g(x) + x^\lambda + \cdots + e^{mx}$ and $f(x) = l(x) + x^n e^{mx}$, then our particular solution is $y_p = h(x) + x^{\lambda+1}(A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0) e^{mx}$.

The idea for the repeated cases is to get rid of all the repeats while preserving the same amount of constants. The cases that are outlined above are very general, so I made a table of the type of expressions we would most likely come across

Case	Characteristic Solution	Repeat	Form of Particular Solution
Case 2	$y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	$x^n e^{r_1 x}$	$y_p = x(A_n x^n + \cdots + A_x + A_0) e^{r_1 x}$
	$y_c =^{\xi x} (A\cos(\theta x) + B\sin(\theta x))$	$x^n e^{\xi x}\cos(\theta x)$	$y_p = x(A_n x^n + \cdots + A_x + A_0) e^{\xi x}\cos(\theta x)$
Case 3	$y_c = (c_1 + c_2 x) e^{\lambda x}$	$x^n e^{\lambda x}$	$y_p = x^2(A_n x^n + \cdots + A_x + A_0) e^{\lambda x}$

It can be tricky to figure out what y_p has to be in the beginning, but hopefully some practice problems will help us.

6) $r^2+2r=r(r+2)=0 \Rightarrow r=0,-2$, so $y_c=c_1+c_2e^{-2t}$. Since $f(t)=3+4\sin 2t$, our initial guess for the particular solution is $y_p=A+B\cos 2t+C\sin 2t$, but this would be incorrect because we already have a lone constant in our characteristic solution, so our actual particular solution is $y_p=At+B\cos 2t+C\sin 2t$. Plugging this into the ODE gives,

$$4(C-B)\cos 2t-4(B+C)\sin 2t+2A=3+4\sin 2t.$$

Matching the terms gives $2A=3\Rightarrow A=3/2$ immediately. From the cosine term we get $4(C-B)=0\Rightarrow C=B$ because there is no cosine term on the right hand side. From the sine terms we have $-4(B+C)=8B=4\Rightarrow C=B=-1/2$, so our particular solution is $y_p=\frac{3}{2}t-\frac{1}{2}\cos 2t-\frac{1}{2}\sin 2t$. Then our general solution is

$$y=c_1+c_2e^{-2t}+\frac{3}{2}t-\frac{1}{2}\cos 2t-\frac{1}{2}\sin 2t.$$

7) $r^2+9=0\Rightarrow r=\pm 3i$, then $y_c=A\cos 3t+B\sin 3t$. Since $f(t)=t^2e^{3t}+6$, $y_p=(At^2+Bt+C)e^{3t}$, and there are no repeats. Plugging this into the ODE gives

$$\begin{aligned} 2Ae^{3t}+6(2At+B)e^{3t}+18(At^2+Bt+C)e^{3t}+9D&=t^2e^{3t}+6\\ \Rightarrow 18At^2e^{3t}+(12A+18B)te^{3t}+(2A+6B+18C)e^{3t}+9D&=t^2e^{3t}+6. \end{aligned}$$

Matching terms immediately gives us $9D=6\Rightarrow D=2/3$. From the t^2e^{3t} we get $18A=1\Rightarrow A=1/18$. The other terms are zero so we get, $12/18+18B=0\Rightarrow B=-1/27$, and $1/9+2/9+18C=0\Rightarrow C=1/162$. So, our particular solution is $y_p=(t^2/18-t/27+1/162)e^{3t}+2/3$. Then our general solution is

$$y=A\cos 3t+B\sin 3t+\left(\frac{1}{18}t^2-\frac{1}{27}t^2+\frac{1}{162}\right)e^{3t}+\frac{2}{3}.$$

18) $r^2-2r-3=(r-3)(r+1)=0\Rightarrow r=3,-1\Rightarrow y_c=c_1e^{3t}+c_2e^{-t}$. Since $f(t)=3te^{2t}$, $y_p=(At+B)e^{2t}$ and there are no repeats. Plugging this into the ODE gives

$$4Ae^{2t}+4(At+B)e^{2t}-2Ae^{2t}-4(At+B)e^{2t}-3(At+B)e^{2t}=-3Ate^{2t}+(2A-3B)e^{2t}=3te^{2t}.$$

Matching the te^{2t} gives $-3A=3\Rightarrow A=-1$. The other term is zero, so we get $-2-3B=0\Rightarrow B=-2/3$. This gives us $y_p=(-t-2/3)e^{2t}$, then our general solution is

$$y=c_1e^{3t}+c_2e^{-t}+\left(-t-\frac{2}{3}\right)e^{2t}.$$

The first initial condition gives $y(0)=c_1+c_2-2/3=1\Rightarrow c_1+c_2=5/3$, and he second gives $y'(0)=3c_1-c_2-1-4/3=0\Rightarrow 3c_1-c_2=7/3$. Now we add the equations to get $4c_1=4\Rightarrow c_1=1\Rightarrow c_2=2/3$. Then our solution is

$$y=e^{3t}+\frac{2}{3}e^{-t}+\left(-t-\frac{2}{3}\right)e^{2t}.$$

24) For this problem we only need the form of the particular solution. In order to get that we still have to compute the characteristic solution: $r^2+2r+2=0\Rightarrow r=-1\pm i$, which gives $y_c=e^{-t}(c_1\sin t+c_2\cos t)$. From $f(x)$ we can guess a particular solution of

$$\begin{aligned} y_p&\stackrel{?}{=}e^{-t}[A+B\cos t+C\sin t+(D_2t^2+D_1t+D_0)\cos t+(E_2t^2+E_1t+E_0)\sin t]\\ &\stackrel{?}{=}e^{-t}[A+(B_2t^2+B_1t+B_0)\cos t+(C_2t^2+C_1t+C_0)\sin t] \end{aligned}$$

However, this would be wrong due to the repeats. So, we need to multiply the cosine and sine block out by t

$$y_p=e^{-t}[A+t(B_2t^2+B_1t+B_0)\cos t+t(C_2t^2+C_1t+C_0)\sin t]$$

3.6 Variation of Parameters

Consider the ODE

$$y'' + p(x)y' + q(x)y = f(x) \tag{3}$$

and suppose we have the following characteristic solution

$$y_c = c_1 y_1 + c_2 y_2 \tag{4}$$

What if for the full solution to (3) we can think of the "constants" as functions; i.e. $y = u_1(x)y_1 + u_2(x)y_2$. We can use this as an ansatz and plug it into the ODE. For the derivative we get

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

We only want one derivative in our final equation so lets force

$$u_1' y_1 + u_2' y_2 = 0 \tag{5}$$

so $y' = u_1 y_1' + u_2 y_2'$, then

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Plugging this into 3 gives

$$u_1 y_1' + u_2' y_2' + u_1[\cancelto{0}{y_1'' + p(x)y_1' + q(x)y_1}] + u_2[\cancelto{0}{y_2'' + p(x)y_2' + q(x)y_2}] = f(x)$$

Notice that the terms in brackets cancel because they are solutions to the nonhomogeneous ODE. This gives us our second equation

$$u_1' y_1' + u_2' y_2' = f(x) \tag{6}$$

From (5) we get $u_1' = -u_2' y_2 / y_1$. We plug this into 6 in order to get an expression for u_2

$$-u_2' y_1' \frac{y_2}{y_1} + u_2' y_2' = f(x) \Rightarrow -u_2' y_1' y_2 + u_2' y_2' y_1 = f(x) y_1 \Rightarrow u_2' = \frac{f(x) y_1}{y_2' y_1 - y_1' y_2} = \frac{f(x) y_1}{W(y_1, y_2)}.$$

Now we plug this into our expression for u_1 to get

$$u_1' = -\frac{f(x) y_2}{W(y_1, y_2)}$$

Then we integrate to get

$$u_1 = -\int \frac{f(x) y_2}{W(y_1, y_2)} dx \tag{7}$$

$$u_2 = \int \frac{f(x) y_1}{W(y_1, y_2)} dx \tag{8}$$

Then plugging back into our original ansatz gives us

$$y = -y_1 \int \frac{f(x) y_2}{W(y_1, y_2)} dx + y_2 \int \frac{f(x) y_1}{W(y_1, y_2)} dx$$

Theorem 1. *Suppose the ODE* (3) *has a unique solution on I open. Assume it has the characteristic solution* (4)*. Then*

$$y = -y_1 \int \frac{f(x) y_2}{W(y_1, y_2)} dx + y_2 \int \frac{f(x) y_1}{W(y_1, y_2)} dx \tag{9}$$

is the general solution.

Now we could just use this theorem for all our problems. The only downfall is that we will have to memorize this formula. So, just in case you forget the formula, do know how to work out the derivation, and try to use the derivation on specific problems.

2) $r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow y_c = c_1 e^{2t} + c_2 e^{-t}$, so $y_1 = e^{2t}$ and $y_2 = e^{-t}$. First we calculate the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -3e^t.$$

Now lets compute our two integrals separately

$$\int \frac{f(t)y_2}{W(y_1,y_2)}dt = \int \frac{e^{-t}\cdot 2e^{-t}}{-3e^t}dt = -\frac{2}{3}\int e^{-3t}dt = -\frac{2}{9}e^{-3t} + c_3.$$

and

$$\int \frac{f(t)y_1}{W(y_1,y_2)}dt = \int \frac{e^{2t}\cdot 2e^{-t}}{-3e^t}dt = \frac{2}{3}\int dt = -\frac{2}{3}t + c_4.$$

Then plugging this back into (9) gives

$$y = -e^{2t}\left[-\frac{2}{9}e^{-3t} + c_3\right] + e^{-t}\left[\frac{2}{3}t + c_4\right] = \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t} - c_3e^{2t} + c_4e^{-t} = c_5e^{-t} - \frac{2}{3}te^{-t} - c_3e^{2t}.$$

10) $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow y_c = c_1e^t + c_2te^t$, then $y_1 = e^t$ and $y_2 = te^t$, then we compute the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}.$$

Now we compute the two integrals

$$\int \frac{f(t)y_2}{W(y_1,y_2)}dt = \int \frac{te^t\cdot e^t/(1+t^2)}{e^{2t}}dt = \int \frac{tdt}{1+t^2} = \frac{1}{2}\ln(1+t^2) + c_3.$$

and

$$\int \frac{f(t)y_1}{W(y_1,y_2)}dt = \int \frac{e^t\cdot e^t/(1+t^2)}{e^{2t}}dt = \int \frac{dt}{1+t^2} = \tan^{-1}t + c_4$$

Plugging this into (9) gives

$$y = -\frac{1}{2}e^t\ln(1+t^2) + te^t\tan^{-1}t - c_3e^t + c_4te^t.$$

11) $r^2 - 5r + 6 = (r-3)(r-2) = 0 \Rightarrow y_c = c_1e^{3t} + c_2e^{2t}$, so $y_1 = e^{3t}$ and $y_2 = e^{2t}$. The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = -e^{5t}$$

Plugging this into (9) gives us

$$y = -e^{3t}\int \frac{e^{2t}g(t)}{-e^{5t}}dt + e^{2t}\int \frac{e^{3t}g(t)}{-e^{5t}}dt = e^{3t}\int e^{-3t}g(t)dt - e^{2t}\int e^{-2t}g(t)dt$$

14) We must first convert this into standard form

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t.$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t.$$

Then we plug into (9) to get

$$y = -t\int \frac{te^t\cdot 2t}{t^2e^t}dt + te^t\int \frac{t\cdot 2t}{t^2e^t}dt = -t\int 2dt + te^t\int 2e^{-t}dt = -2t^2 + c_1tc_2te^t.$$

So the particular solution is

$$y_p = -2t^2$$

20) We convert this to standard form

$$y'' + \frac{1}{x}y' + \frac{x^2 - 0.25}{x^2}y = \frac{g(x)}{x^2}.$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} x^{-1/2}\sin x & x^{-1/2}\cos x \\ -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x & -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x \end{vmatrix} = -\frac{1}{x}$$

Then plugging into (9) gives

$$\begin{aligned} y &= -x^{-1/2}\sin x \int \frac{x^{-1/2}\cos x \cdot g(x)/x^2}{-1/x}dx + x^{-1/2}\cos x \int \frac{x^{-1/2}\sin x \cdot g(x)/x^2}{-1/x}dx \\ &= x^{-1/2}\sin x \int \frac{\cos x g(x)}{x\sqrt{x}}dx - x^{-1/2}\cos x \int \frac{\sin x g(x)}{x\sqrt{x}}dx. \end{aligned}$$