6.6 CONVOLUTIONS

To derive this we need knowledge of Calc III, which isn't a prerequisite, so we will just define it. A convolution is the following operator,

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau. \tag{1}$$

The Laplace Transform is as follows,

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \tag{2}$$

It should be noted that this is similar to multiplication and has some of the same properties:

$$1)\ f * g = g * f \qquad 2)\ f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad 3)\ (f * g) * h = f * (g * h)$$

Now lets do some problems,

7) We take the Laplace Transform of sine and cosine and then multiply them together,

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1},\ \mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1} \Rightarrow \mathcal{L}\{f(t)\} = \frac{s}{(s^2 + 1)^2}.$$

11) Here we take the inverse. We know the transform of sine from above and the inverse transform of $G(s)$. So we get,

$$\mathcal{L}^{-1}\{F(s)\} = \int_0^t \sin(\tau)g(t - \tau)d\tau$$

17) Here we take the Laplace Transform of the entire IVP,

$$-\underset{-3}{y'(0)} - \underset{2}{sy(0)} + s^2 Y - \underset{2}{4y(0)} + 4sY + 4Y = G(s) \Rightarrow (s^2 + 4s + 4)Y = 2s + 5 + G(s) \Rightarrow Y = \frac{2s + 5}{(s + 2)^2} + \frac{G(s)}{(s + 2)^2}.$$

We employ partial fractions,

$$\frac{A}{s + 2} + \frac{B}{(s + 2)^2} = \frac{2s + 5}{(s + 2)^2} \Rightarrow As + 2A + B = 2s + 5.$$

This gives, $A = 2$, $B = 1$. Then we get,

$$Y = \frac{2}{s + 2} + \frac{1}{(s + 2)^2} + \frac{G(s)}{(s + 2)^2}.$$

Taking the inverse transform gives,

$$y = 2e^{-2t} + te^{-2t} + \int_0^t \tau e^{-2\tau}g(t - \tau)d\tau$$

16) Again,

$$-\underset{-1}{y'(0)} - \underset{1}{sy(0)} + s^2 Y - \underset{1}{y(0)} + sY + \frac{5}{4}Y = \frac{1}{s} - \frac{1}{s}e^{-\pi s} \Rightarrow (s^2 + s + 5/4)Y = s + \frac{1}{s} - \frac{1}{s}e^{-\pi s}$$

$$\Rightarrow Y = \frac{s}{s^2 + s + 5/4} + \frac{1 - e^{-\pi s}}{s(s^2 + s + 5/4)} = \frac{s + 1/2}{(s + 1/2)^2 + 1} - \frac{1/2}{(s + 1/2)^2 + 1} + \frac{1}{(s + 1/2)^2 + 1} \cdot \frac{1 - e^{-\pi s}}{s}$$

$$\Rightarrow y = e^{-t/2}\cos t - \frac{1}{2}e^{-t/2}\sin t + \int_0^t e^{-\tau/2}\sin(\tau)(1 - u_\pi(t - \tau))d\tau.$$

7.1 INTRODUCTION TO SYSTEMS OF FIRST ORDER ODES

In class we went through the example of a simple pendulum. I won't redo that here, but what we take from it is that the pendulum is governed by the ODE, $\theta'' + (g/L)\sin\theta = 0$. And we can convert this into a system of two first order ODEs by letting $\omega = \theta'$, then $\theta' = \omega$ and $\omega' = -(g/L)\sin\theta$. By doing this we could extract a lot of necessary information to an otherwise unsolvable problem (with the methods we know, and in fact we can't find an exact solution). We can use this trick for other problems as done below,

    1) Let $v = u'$, then $v' = -v/2 + 2u$.
    3) Let $v = u'$, then $v' = -v/t + (1/4 - t^2)u/t^2$.
    6) Let $v = u'$, then $v' = g(t) - p(t)v - q(t)u$ and $u(0) = u_0$, $v(0) = u_0'$.
    10) Notice $x_2 = (x_1 - x_1')/2$, then

$$((x_1 - x_1')/2)' = 3x_1 - 4((x_1 - x_1')/2) \Rightarrow x_1' - x_1'' = 2x_1 + 4x_1' \Rightarrow x_1'' + 3x_1' + 2x_1 = 0;\ x(0) = -1,\ x'(0) = -5.$$

Now we solve for x_1, $r^2 + 3r + 2 = (r+2)(r+1) = 0 \Rightarrow r = -2, -1$, then

$$x_1 = c_1 e^{-t} + c_2 e^{-2t}.$$

For the initial conditions we get $x_1(0) = c_1 + c_2 = -1$ and $x_1'(0) = -c_1 - 2c_2 = -5$, then $c_2 = 6$, $c_1 = -7$. Now to solve for x_2 we plug x_1 into the first equation where we have x_2 as a function of x_1 and x_1' to get,

$$x_1 = 6e^{-t} - 7e^{-2t};\ x_2 = -7e^{-t} + 9e^{-2t}.$$

7.2 MATRICES AND 7.3 EIGENVALUES/EIGENVECTORS

We went through these kind of quickly, but know how to multiply vectors by matrices, take the determinant, find whether or not vectors are linearly independent, and find eigenvalues and eigenvectors. Also, you only have to worry about 2x2 matrices.

7.5 HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

These are basically like our second order problems, we just have to take the eigenvalue of the matrix and treat them as our roots. Then we compute the eigenvectors. Recall we find the eigenvalues by computing what values of λ satisfy $\det(A - \lambda I) = 0$, and the eigenvectors are found by computing the values of x that satisfy $(A - \lambda I)x = 0$.

    1) The eigenvalues are

$$\begin{vmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{vmatrix} = (3 - \lambda)(-2 - \lambda) + 4 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 2$$

Now we compute the eigenvectors,

$$\begin{pmatrix} 3 - \lambda_1 & -2 \\ 2 & -2 - \lambda_1 \end{pmatrix} x^{(1)} = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} x^{(1)} = 0 \Rightarrow x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 - \lambda_2 & -2 \\ 2 & -2 - \lambda_2 \end{pmatrix} x^{(2)} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} x^{(2)} = 0 \Rightarrow x^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Then our solution becomes,

$$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

Now, if $c_2 = 0$, $x \to 0$ and if $c_2 \neq 0$, $x \to \infty$.

3) Again we find the eigenvalues,

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1.$$

The eigenvectors are,

$$x^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \ x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then the solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

Here if $c_2 = 0$, $x \to 0$ and if $c_2 \neq 0$, $x \to \infty$.

5) Again,

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = (\lambda+1)(\lambda+3) = 0 \Rightarrow \lambda = -1, \ -3.$$

And the eigenvectors are,

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ x^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Our solution is,

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$$

Here $x \to 0$.

8) Again the eigenvalues are

$$\begin{vmatrix} 3-\lambda & 6 \\ -1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 6 = -6 - \lambda + \lambda^2 + 6 = \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0, \ 1$$

with the eigenvectors,

$$x^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ x^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Then our solution is

$$x = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

So our solution behaves as follows: if $c_2 = 0$, $x = c_1(-2, 1)$; i.e. the first eigenvector. If $c_2 \neq 0$, $x \to \infty$.