3.2 Existence and Uniqueness and the Wronskian

Last time we discussed ODEs of the form,

$$p_n(x)y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \cdots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = 0.$$

Now lets look at the general case of,

$$p_n(x)y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \cdots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = g(x).$$

Lets put this in standard form by dividing through by $p_n(x)$ and naming the new functions "q" and "f",

$$y^{(n)}(x) + q_{n-1}(x)y^{(n-1)}(x) + \cdots + q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = f(x). \tag{1}$$

Consider the simple ODE,

$$y' + q(x)y = f(x);\ q(x) = \begin{cases} 1 \text{ if } x \text{ is irrational,} \\ 0 \text{ if } x \text{ is rational;} \end{cases}$$

In order to solve this we would need to use integrating factors, however notice that q is not integrable (in the usual fashion), so we can't solve this - in fact it has no unique solution. So, we need conditions on $q's$ and f to guarantee that we can find a unique solution. We outline this in the next theorem, however one should proceed with caution because this only works for linear ODEs.

Theorem 1. *Consider ODE (1) with initial conditions:* $y(x_0) = a_0$, $y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$. *Then, if* $q_{n-1}, q_{n-2}, \ldots, q_2, q_1, q_0$ *are continuous on a common interval* I *containing* x_0*, the IVP has exactly one solution on* I*.*

Now we proceed to defining certain important ideas that we will use in our following theorems.

Definition 1. The set of functions $\{h_1, h_2, \ldots, h_{n-1}, h_n\}$ are said to be linearly independent if $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n \neq 0$, otherwise it is said to be linearly dependent.

Definition 2. The expression $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n$ is said to be a linear combination of $h_1, h_2, \ldots, h_{n-1}, h_n$.

Last time we talked about superposition. We will pose it more rigorously in the next theorem. First consider the homogeneous ODE in standard form,

$$y^{(n)}(x) + q_{n-1}(x)y^{(n-1)}(x) + \cdots + q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = 0. \tag{2}$$

Theorem 2. *If* $y_1, y_2, \ldots, y_{n-1}, y_n$ *are solutions to (2), then any linear combination of* y*'s are also solutions.*

For example, $y = c_1y_1 + c_2y_2$, $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, etc. are also solutions.
Now we define what the Wronskian is, which will be a major part of this section.

Definition 3. Suppose $h_1(x), h_2(x), \ldots, h_{n-1}, h_n$ are functions with $n-1$ derivatives, then the Wronskian is defined to be the following determinant,

$$W = \begin{vmatrix} h_1 & h_2 & \cdots & h_n \\ h_1' & h_2' & \cdots & h_n' \\ h_1'' & h_2'' & \cdots & h_n'' \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(n-1)} & h_2^{(n-1)} & \cdots & h_n^{(n-1)} \end{vmatrix} \tag{3}$$

Theorem 3. *Suppose* $y_1, y_2, \ldots, y_{n-1}, y_n$ *are solutions to (2) on* I*, with the usual initial conditions, then* $W \neq 0$ *guarantees they are linearly independent on* $\in I$*.*

So the rewording of the above theorem implies that if the Wronskian is zero at a single point then the function may still be linearly independent.

The next definition and theorem will allow us to find guaranteed linearly independent solutions, but note that these are not necessarily the only linearly independent solutions.

Definition 4. The set of all linearly independent solutions of an ODE is called the <u>fundamental set</u> of that ODE.

For the remaining theorems consider the second order ODE,

$$y'' + q_1(x)y' + q_0(x)y = 0. \tag{4}$$

Theorem 4. *Consider ODE (4), and let y_1, y_2 solve (4) for $x \in I$ such that $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x_0) = 0$, $y_2'(x_0) = 1$. Then, y_1, y_2 form a fundamental set of (4).*

The following theorem is a theorem we use in section 3.3.

Theorem 5. *If $y = u(t) + iv(t)$ solves (4) on I, then so does u and v independently, i.e. if $y = c_1u + ic_2v$ is a solution, so is $y = c_3u + c_4v$.*

The next theorem gives us a formula to compute the Wronskian without having to take a determinant, but it only works for second order ODEs.

Theorem 6 (Abel)**.** *The Wronskian of y_1, y_2 for (4) can be written as,*

$$W(y_1, y_2) = c\exp\left(-\int q_1(x)dx\right), \tag{5}$$

and is zero (if $c = 0$) or nonzero (if $c \neq 0$) for all $x \in I$.

Now lets do some example problems,

1) The derivatives are $2e^{2t}$ and $(-3/2)e^{-3t/2}$, so our Wronskian is,

$$W = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{3}{2}e^{2t-3t/2} - 2e^{2t-3t/2}.$$

3) The derivatives are $-2e^{-2t}$ and $e^{-2t} - 2te^{-2t}$, so our Wronskian is,

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}.$$

9) We put the ODE in standard form,

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}.$$

Notice, that this has discontinuities at $t = 0, 4$, and since we need to include the initial condition, the largest domain where a unique solution exists is $t \in (0, 4)$.

11) Again we convert the ODE into standard form,

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$$

This is discontinuous when $x = 0, 3$, so our largest domain where a unique solution containing the initial condition exists is $x \in (0, 3)$.

17) Here we have an inverse problem. We need to find a g that satisfies the Wronskian given, so lets take the Wronskian and see what we get,

$$W = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}g' - 2e^{2t}g = e^{2t}(g' - 2g) = 3e^{4t} \Rightarrow g' - 2g = 3e^{2t}$$

So we have to solve this first order ODE via integrating factor,

$$\mu = \exp\left(-\int^t 2d\tau\right) \Rightarrow \int d(e^{-2t}g) = \int 3dt \Rightarrow e^{-2t}g = 3t + C \Rightarrow g = 3te^{2t} + Ce^{2t}.$$

23) We go straight to the characteristic polynomial, $r^2+4r+3=(r+1)(r+3)=0 \Rightarrow r=-1,-3$, so our general solution is $y=c_1e^{-x}+c_2e^{-3x}$. Now, by Theorem 4, we solve two different IVPs for this ODE: $y_1(1)=c_1e^{-1}+c_2e^{-3}=1$ and $y_1'(1)=-c_1e^{-1}-3c_2e^{-3}=0$. By summing the two equations we get $-2c_2e^{-3}=1 \Rightarrow c_2=-e^3/2$, this gives $c_1=3e/2$, so our first solution is $y_1=\frac{3}{2}e^{(1-x)}-\frac{1}{2}e^{3(1-x)}$. For the second solution we have $y_2(1)=c_1e^{-1}+c_2e^{-3}=0$ and $y_2'(1)=-c_1e^{-1}-3c_2e^{-3}$. We easily get $c_2=-e^3/2$ and then $c_1=e/2$, which gives us a solution of $y_2=\frac{1}{2}e^{(1-x)}-\frac{1}{2}e^{3(1-x)}$. So, the following equations make a fundamental set of the ODE,

$$y_1=\frac{3}{2}e^{(1-x)}-\frac{1}{2}e^{3(1-x)};\ y_2=\frac{1}{2}e^{(1-x)}-\frac{1}{2}e^{3(1-x)}.$$

27) For the first solution we have $y_1'=1 \Rightarrow y_1''=0 \Rightarrow -xy_1'+y_1'=0$. For the second solution we have $y_2'=\cos x \Rightarrow y_2''=-\sin x$, then $(1-x\cot x)(-\sin x)-x\cos x+\sin x=-\sin x+x\cos x-x\cos x+\sin x=0$. Now, we take the Wronskian of these,

$$W=\begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}=x\cos x-\sin x\neq 0 \text{ for } x\in(0,\pi).$$

So, they are linearly independent on that domain.

30) We put the ODE into standard form: $y''+(\tan t)y'-ty/\cos t=0$. Then we use Abel's theorem to get,

$$W=c\exp\left(-\int(\tan t)dt\right)=c\cos t.$$

19) Recall the Wronskians in 2D is $W(f,g)=fg'-f'g$ and $W(u,v)=uv'-u'v$. Then

$$\begin{aligned}W(u,v)&=(2f-g)(f'+2g')-(2f'-g')(f+2g)=2ff'+4fg'-gf'-2gg'-2f'f-4f'g+g'f+2g'g\\&=4[fg'-f'g]+[fg'-f'g]=5W(f,g).\end{aligned}$$

3.3 Complex Roots

Again consider the ODE,

$$ay''+by'+cy=0, \tag{6}$$

which has the characteristic polynomial equation,

$$ar^2+br+c=0. \tag{7}$$

Using the quadratic formula we get,

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

What if $b^2-4ac<0$? Then r is of the form $r=\xi\pm i\theta$ where $\xi,\theta\in\mathbb{R}$, but this means r is a complex conjugate. However, we do the same thing as usual to get,

$$y=c_1e^{r_1x}+c_2e^{r_2x}=c_1e^{(\xi+i\theta)x}+c_2e^{(\xi-i\theta)x}=e^{\xi x}\left(c_1e^{i\theta x}+c_2e^{-i\theta x}\right).$$

We need to deal with the part inside the parentheses, and we do this by what's called, *Euler's Identity.* And we can derive this fairly easily by using Taylor series, since we know the taylor series,

$$e^{it}=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^nt^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}\frac{(-1)^{n+1}t^{2n+1}}{(2n+1)!}=\cos t+i\sin t. \tag{8}$$

Then our solution becomes,

$$y=e^{\xi x}[c_1(\cos\theta x+i\sin\theta x)+c_2(\cos\theta x-i\sin\theta x)]=e^{\xi x}[(c_1+c_2)\cos\theta x+i(c_1-c_2)\sin\theta x].$$

However, we only want real solutions. Notice that $\cos\theta x$ and $\sin\theta x$, with the proper constant coefficients, are solutions to (6) independently. So, by Theorem 5, $y=e^{\xi x}(A\cos\theta x+B\sin\theta x)$ is also a solution. We have just developed a theorem,

Theorem 7. *If (7) has complex roots, i.e. $r = \xi + i\theta$, then the general solution of (6) is,*

$$y = e^{\xi x}(A\cos\theta x + B\sin\theta x). \tag{9}$$

Now, lets do some examples,

4) Applying Euler's identity, $e^{2-i\pi/2} = e^2(\cos\pi/2 - i\sin\pi/2) = -ie^2$.

6) $\frac{1}{\pi}e^{i2\ln\pi} = \frac{1}{\pi}(\cos(2\ln\pi) + i\sin(2\ln\pi))$.

10) We go to the characteristic polynomial, $r^2 + 2r + 2 = 0$ and use the quadratic formula, $r = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$, which gives us a general solution of $y = e^{-t}(A\cos t + B\sin t)$.

18) Again our characteristic polynomial gives, $r^2 + 4r + 5 = 0$, and the quadratic formula gives, $r = (-4 \pm \sqrt{-4})/2 = -2 \pm i$, so our general solution is $y = e^{-2t}(A\cos t + B\sin t)$. Now we go to our initial conditions: $y(0) = A = 1$. Then, $y'(t) = -2e^{-2t}(\cos t + B\sin t) + e^{-2t}(-\sin t + B\cos t)$, so $y'(0) = -2 + B = 0 \Rightarrow B = 2$. So, our solution is $y = e^{-2t}(\cos t + 2\sin t)$.

20) As per usual we have $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y = A\cos t + B\sin t$. From the first initial condition we have, $y(\pi/3) = A/2 + \sqrt{3}B/2 = 2 \Rightarrow A = 4 - \sqrt{3}B$. From the second initial condition we have, $y'(\pi/3) = -\sqrt{3}A/2 + B/2 = -4 \Rightarrow -\sqrt{3} + 3B/2 + B/2 = -2\sqrt{3} + 2B = -4 \Rightarrow B = \sqrt{3} - 2$. So, we get $A = 1 + 2\sqrt{3}$. Then our solution is $y = (1 + 2\sqrt{3})\cos t + (\sqrt{3} - 2)\sin t$.

3.4 Repeated Roots and Reduction of Order

Repeated Roots: Again consider a second order homogeneous IVP with it's respective characteristic polynomial equation,

$$y'' + by' + cy = 0;\ y(0) = A, y'(0) = B \tag{10}$$

$$r^2 + br + c = 0 \tag{11}$$

Then our roots (also called eigenvalues) are $r = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$. What if $b^2 - 4c = 0 \Rightarrow c = b^2/4$? Then $r_{1,2} = -b/2$. If we plug this in as usual we get $y = c_1e^{-bx/2} + c_2e^{-bx/2} = (c_1 + c_2)e^{-bx/2}$. However, this only gives us one constant so there is no way we can satisfy the two initial conditions. So, we need another solution in addition to the one we have.

Suppose the "constant" $c_1 + c_2$ is not a constant, but rather a function of x; i.e. $y = v(x)e^{-bx/2}$. We have to figure out if a v will satisfy our ODE, and if so, what v is it. We want to plug into 10. The derivatives are

$$y' = v'(x)e^{-bx/2} - \frac{b}{2}e^{-bx/2}v(x) \Rightarrow y'' = v''(x)e^{-bx/2} - be^{-bx/2}v'(x) + \frac{b^2}{4}e^{-bx/2}v(x).$$

Plugging into the ODE gives

$$e^{-bx/2}\left(v'' + (-b+b)v' + \left(\frac{b^2}{4} - \frac{b^2}{2} + \frac{b^2}{4}\right)\right) = e^{-bx/2}v'' = 0$$

Since $\exp(-bx/2)$ can't be zero in finite x, $v'' = 0 \Rightarrow v' = c_3 \Rightarrow v = c_3x + c_4$, which gives us a solution of

$$y = (c_3x + c_4)e^{-bx/2}.$$

We still don't know if this is a legitimate solution or not yet, but let's write down the theorem anyway and then prove it.

Theorem 8. *Consider the ODE*

$$ay'' + by' + c = 0. \tag{12}$$

If the characteristic polynomial has repeated roots; i.e. $r_{1,2} = \lambda$, then the general solution to 12 is,

$$y = (c_1 + c_2x)e^{\lambda x}. \tag{13}$$

Proof. Clearly 13 is a solution to 12, which we verified by differentiating and plugging into the ODE. Furthermore, $W(e^{\lambda x}, xe^{\lambda x}) = e^{2\lambda x} \neq 0$, which we calculated in class. $\square$

Now, lets solve some problems before moving onto the second part of this section.

2) The characteristic equation is $9r^2+6r+1=0 \Rightarrow r=-1/3$, then our solution is $y=(c_1+c_2x)e^{-x/3}$.

8) As per usual, $16r^2+24r+9=0 \Rightarrow r=-3/4 \Rightarrow y=(c_1+c_2x)e^{-3x/4}$.

15) I'll just show part d here. Solving the ODE gives us $4r^2+12r+9=0 \Rightarrow r=-3/2 \Rightarrow y=(c_1+c_2x)e^{-3x/2}$. The first initial condition gives $y(0)=c_1=1$. The other one gives $y'(0)=-3/2+c_2=b \Rightarrow c_2=b+3/2$. So, when $b<-3/2$ it's eventually negative, but when $b \geq -3/2$ it's always positive.

12) Again, we have to solve an IVP. Our roots are $r^2-6r+9=0 \Rightarrow r=3$. So, our solution is $y=(c_1+c_2x)e^{3x}$. From the initial conditions we have, $y(0)=c_1=0$ and $y'(0)=c_2=2$, so our solution is $y=2xe^{3x}$.

20) We're going to first solve for one solution, get our Wronskian, then use the Wronskian to solve for the other solution,
   (a) $y_1=e^{-at}$
   (b) $W=C\exp\left(-\int 2a dt\right)=Ce^{-2at}$.
   (c) $W=e^{-at}y_2'+ae^{-at}y_2=e^{-2at} \Rightarrow y_2'+ay_2=e^{-at}$. We can solve this via integrating factor: $\mu=e^{at}$, then

$$e^{at}y_2=\int dt=t \Rightarrow y_2=te^{-at}.$$

Reduction of Order: The method we used in the beginning of class is called *reduction of order*, but we'll see that this is far more powerful than it first seemed. Consider the ODE

$$y''+p(x)y'+q(x)y=0 \tag{14}$$

and suppose we know one solution, say $y=y_1(x)$, then we "guess" the full solution is of the form $y=v(x)y_1(x)$. First we find the derivatives

$$y'=y_1'v+v'y_1 \Rightarrow y''=y_1''v+2v'y_1'+v''y_1.$$

Plugging this in and grouping the respective v's gives us

$$\begin{aligned} y_1''v+2v'y_1'+v''y_1+py_1'v+pv'y_1+qy_1v &= y_1v''+(2y_1'+py_1)v'+\cancelto{0}{(y_1''+py_1'+qy_1)}v \\ &= y_1v''+(2y_1'+py_1)v'=0. \end{aligned}$$

And set $u=v'$ to get

$$\begin{aligned} &y_1u'+(2y_1'+py_1)u=0 \Rightarrow u'=-\frac{2y_1'+py_1}{y_1}u=0 \Rightarrow \int\frac{du}{u}=-\int\frac{2y_1'+py_1}{y_1}dx \\ &\Rightarrow \ln u=-\int\frac{2y_1'+py_1}{y_1}dx \Rightarrow u=\exp\left(-\int\frac{2y_1'+py_1}{y_1}dx\right) \\ &\Rightarrow v=\int\exp\left(-\int\frac{2y_1'+py_1}{y_1}dx\right). \end{aligned}$$

Now, we'll do some problems

27) Let $y = vy_1 \Rightarrow x(v''y_1 + 2v'y_1' + vy_1'') - (v'y_1 + vy_1') + 4x^3y_1 = 0$. Grouping all the v, v', and v'' terms gives

$$xy_1v'' + 2xy_1'v' - y_1v' + \underbrace{(xy_1'' - y_1' + 4x^3y_1)}_{0}v = xy_1v'' + 2xy_1'v' - y_1v' = 0.$$

Set $u = v'$, then

$$u' + \left(\frac{2y_1'}{y_1} - \frac{1}{x}\right)u = 0 \Rightarrow u' = \left(\frac{1}{x} - \frac{4x\cos x^2}{\sin x^2}\right)u$$

$$\Rightarrow \ln u = \ln x - 2\int \frac{2x\cos x^2}{\sin x^2}dx = \ln x - \ln\sin^2 x^2 + C_0 \Rightarrow u = \frac{kx}{\sin^2 x^2}$$

$$\Rightarrow v = k\int x\csc^2 x^2 dx = k_1\cot x^2 + C_1 \Rightarrow y = k_1\cos x^2 + C_1\sin x^2.$$

So, $y_2 = \cos x^2$.

29) Again we let $y = vy_1 \Rightarrow x^2(v''y_1 + 2v'y_1' + vy_1'') - (x - 0.1875)vy_1 = 0$. Grouping gives

$$x^2y_1v'' + 2x^2y_1'v' + \underbrace{[x^2y_1'' - (x - 0.1875)y_1]}_{0}v = x^2y_1v'' + 2x^2y_1'v' = 0.$$

Set $u = v'$, then

$$u' = -2\frac{y_1'}{y_1}u = \left(\frac{-2}{\sqrt{x}} - \frac{1}{2x}\right)u \Rightarrow \ln u = -2\int x^{-1/2}dx + \frac{1}{2}\int\frac{dx}{x} = -4\sqrt{x} - \frac{1}{2}\ln x + C$$

$$\Rightarrow u = \frac{k}{\sqrt{x}}e^{-4\sqrt{x}} \Rightarrow v = ke^{-4\sqrt{x}} + C \Rightarrow y = kx^{1/4}e^{-2\sqrt{x}} + Cx^{1/4}e^{2\sqrt{x}}.$$

So, $y_2 = x^{1/4}e^{-2\sqrt{x}}$