5.1 Power Series Review

Know it!

5.2 Series Solutions

Consider the ODE

$$y^{(n)}+F_{n-1}(x)y^{(n-1)}+\cdots+F_1(x)y'+F_0(x)y=Q(x). \tag{1}$$

First lets define a few things.

Definition 1. A point $x=x_0$ is said to be an ordinary point of (1) if $F_n,\ldots,F_0,Q$ all have convergent Taylor series in a neighborhood of x_0. However, if at least one function does not satisfy this criterion, $x=x_0$ is called a singular point.

For this section we consider the problem

$$P(x)y''+Q(x)y'+R(x)y=0;\ x=x_0, \tag{2}$$

where $x=x_0$ is an ordinary point and R,Q,P are polynomials. We make the ansatz:

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n \tag{3}$$

$$\Rightarrow y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n \tag{4}$$

$$\Rightarrow y''=\sum_{n=1}^{\infty}n(n+1)a_{n+1}(x-x_0)^{n-1}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n \tag{5}$$

Then we plug this into the ODE (2) and try to solve for the "a's".

Lets do some problems,

3) (a) Plugging into the ODE gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n-(n+1)a_{n+1}x(x-1)^n-a_n(x-1)^n=0.$$

So basically we're stuck, unless ... we let $x=1+(x-1)$. Now we have like terms! Yay!

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n-(n+1)a_{n+1}(x-1)^n-(n+1)a_{n+1}(x-1)^{n+1}-a_n(x-1)^n=0.$$

By matching like terms we get

$$x^0: \qquad 2a_2-a_1-a_0=0\Rightarrow a_2=\frac{1}{2}(a_1+a_0),$$

$$x^m: \qquad (m+2)(m+1)a_{m+2}-(m+1)a_{m+1}-(m+1)a_m=0\Rightarrow a_{m+2}=\frac{a_{m+1}+a_m}{m+2}.$$

Notice that we cannot solve for a_1 and a_0 because these are like our c_1 and c_2 where we have to solve for them using initial conditions, if given.

(b) This means $a_0=0$ gives one solution and $a_1=0$ gives another,

$$a_0=0\Rightarrow a_2=\frac{a_1}{2}\Rightarrow a_3=\frac{a_1}{2}\Rightarrow a_4=\frac{a_1}{4}\cdots\Rightarrow y_2=(x-1)+\frac{1}{2}(x-1)^2+\frac{1}{2}(x-1)^3+\frac{1}{4}(x-1)^4+\cdots$$

$$a_1=0\Rightarrow a_2=\frac{a_0}{2}\Rightarrow a_3=\frac{a_0}{6}\Rightarrow a_4=\frac{a_0}{6}\cdots\Rightarrow y_1=1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{6}(x-1)^4+\cdots$$

6) (a) Plugging into the ODE gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(2+x^2)x^n - (n+1)a_{n+1}x^{n+1} + 4a_n x^n$$
$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + (n+2)(n+1)a_{n+2}x^{n+2} - (n+1)a_{n+1}x^{n+1} + 4a_n x^n = 0.$$

By matching terms we get

$$x^0: \quad 4a_2 + 4a_0 = 0 \Rightarrow a_2 = -a_0$$
$$x^1: \quad 12a_3 + 3a_1 = 0 \Rightarrow a_3 = -\frac{1}{4}a_1$$
$$x^m: \quad 2(m+2)(m+1)a_{m+2} + m(m-1)a_m - ma_m + 4a_m = 0 \Rightarrow a_{m+2} = -\frac{m^2-2m+4}{2(m+2)(m+1)}$$

(b) Now we find the first few terms of our two solutions

$$a_0 = 0 \Rightarrow a_2 = a_4 = \cdots = 0, \text{ so } a_3 = -\frac{1}{4}a_1 \Rightarrow a_5 = \frac{7}{160}a_1 \Rightarrow y_2 = x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \cdots$$
$$a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0, \text{ so } a_2 = -a_0 \Rightarrow a_4 = \frac{1}{6}a_0 \Rightarrow y_1 = 1 - x^2 + \frac{1}{6}x^3 + \cdots$$

11) (a) Plugging into the ODE gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(3-x^2)x^n - 3(n+1)a_{n+1}x^{n+1} - a_n x^n$$
$$= \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n - (n+2)(n+1)a_{n+2}x^{n+2} - 3(n+1)a_{n+1}x^{n+1} - a_n x^n = 0$$

Matching terms gives

$$x^0: \quad 6a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{6}a_0$$
$$x^1: \quad 18a_3 - 3a_1 - a_1 = 18a_3 - 4a_1 = 0 \Rightarrow a_3 = \frac{2}{9}a_1$$
$$x^m: \quad 3(m+2)(m+1)a_{m+2} - m(m-1)a_m - 3ma_m - a_m = 0 \Rightarrow a_{m+2} = \frac{1+3m+m^2-m}{3(m+2)(m+1)}a_m = \frac{m+1}{3(m+2)}a_m$$

(b) For the first few terms we get

$$a_0 = 0 \Rightarrow a_2 = a_4 = \cdots = 0 \Rightarrow a_3 = \frac{2}{9}a_1 \Rightarrow a_5 = \frac{4}{15}a_3 = \frac{8}{135}a_1 \Rightarrow y_2 = x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \cdots$$
$$a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0 \Rightarrow a_2 = \frac{1}{6}a_0 \Rightarrow a_4 = \frac{3}{12}a_2 = \frac{1}{24}a_0 \Rightarrow y_1 = 1 + \frac{1}{6}x^2 + \frac{1}{24}x^4 + \cdots$$

5.4 Euler's Equation; Regular Singular Points

Consider the ODE

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = 0 \tag{6}$$

This has a singular point because if we put this into standard form we get

$$y'' + \alpha\frac{1}{x}y' + \beta\frac{1}{x^2}y = 0,$$

which violates the existence and uniqueness theorem at $x = 0$. We obviously don't know how to deal with this problem. But there is a similar problem that we do know how to deal with,

$$y''(\xi) + ay'(\xi) + by(\xi) = 0 \tag{7}$$

Basically we need to make a change of variables on x in order to get rid of the $x's$ in the coefficients. What do we know that gives us $1/x$ every time we differentiate? $\xi = \ln x$ does the trick. Taking the derivatives are a little different than what we are used to, but very intuitive due to Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{1}{x}\frac{dy}{d\xi},$$
$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{d\xi}\frac{d\xi}{dx} = \frac{1}{x}\left(e^{-\xi}\frac{dy}{d\xi}\right)' = \frac{1}{x}\left(-e^{-\xi}\frac{dy}{d\xi} + e^{-\xi}\frac{d^2y}{dx^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right)$$

Plugging this back into (6) gives us

$$\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi} + \alpha\frac{dy}{d\xi} + \beta y = y'' + ay' + by = 0$$

To solve (7) we use the ansatz $y = \exp(r\xi)$, so to solve (6) we use $y = x^r$. Lets think of a slightly more general second order ODE for this part

$$Ax^2y'' + Bxy' + Cy = 0$$

Then plugging into this gives

$$Ax^2[r(r-1)]x^{r-2} + Bxrx^{r-1} + Cx^r = Ar(r-1)x^r + Brx^r + Cx^r = 0 \Rightarrow Ar(r-1) + Br + C = 0.$$

This is our characteristic polynomial of Euler's equation. And we have the usual cases:

Cases	Solution	Comment
Distinct Roots	$y = c_1x^{r_1} + c_2x^{r_2}$	
Repeated Roots	$y = (c_1 + c_2\ln\lvert x\rvert)x^r$	because $\xi = \ln x$
Complex Conjugate Roots	$y = x^\lambda(A\cos(\mu\ln x) + B\sin(\mu\ln x))$	where $r = \lambda \pm i\mu$

Now lets do some problems

5) The characteristic polynomial is $r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0$, so we have repeated roots $r = 1$, then $y = (c_1 + c_2\ln|x|)x;\ x \neq 0$.
12) The characteristic polynomial is $r(r-1) - 4r + 4 = r^2 - 5r + 4 = (r-1)(r-4) = 0$, then $y = c_1x + c_2x^4;\ x \neq 0$.
11) The characteristic polynomial is $r(r-1) + 2r + 4 = r^2 + r + 4 = 0$, then $r = (-1 \pm i\sqrt{5})/2$, so

$$y = |x|^{-1/2}\left[A\cos\left(\frac{\sqrt{15}}{2}\ln|x|\right) + B\sin\left(\frac{\sqrt{15}}{2}\ln|x|\right)\right].$$

There is one more small theoretical thing we have to discuss.

Definition 2. Suppose $x = x_0$ is a singular point of the ODE

$$y'' + P(x)y' + Q(x)y = 0$$

If $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ have convergent Taylor series at $x = x_0$, then x_0 is called a regular singular point. Otherwise it is called an irregular singular point.

Lets do one example of this

19) We convert this to standard form

$$y'' + \frac{x-2}{x^2(1-x)}y' - \frac{3}{x(1-x)}y = 0$$

So our singular points are $x = 0, 1$. Since these are polynomials it suffices to take the limit and see if it converges,

$$x = 0: \qquad \lim_{x\to 0} xP(x) = \lim_{x\to 0} x\frac{x-2}{x^2(1-x)} = \lim_{x\to 0}\frac{x-2}{x(1-x)} = \pm\infty$$

So, $x = 0$ is an irregular singular point.

$$\lim_{x\to 1}(x-1)P(x) = \lim_{x\to 1}(x-1)\frac{x-2}{x^2(1-x)} = \lim_{x\to 1}\frac{2-x}{x} = 1\checkmark$$

$$\lim_{x\to 1}(x-1)^2Q(x) = \lim_{x\to 1}(x-1)^2\frac{-3}{x(1-x)} = \lim_{x\to 1}\frac{3(1-x)}{x} = 0\checkmark$$

This means that $x = 1$ is a regular singular point.

5.5 Series Solutions; around regular singular points

In section 5.2 we did series solutions around ordinary points and in 5.4 we did simple solutions around regular singular points. The ansatz we used in 5.4, in essence, kills off the singularities and allows us to solve the problem. Lets use these two principles to solve a problem of the form

$$x^2y'' + xP(x)y' + Q(x)y = 0;\ x_0 = 0, \tag{8}$$

or more generally an equation that can be transformed into that form. Lets use the ansatz

$$y = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}$$

$$\Rightarrow y' = \sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}$$

$$\Rightarrow y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2}$$

These problems are quite lengthy, so we'll only do two.

3) (a) We show $x = 0$ is a regular singular point by taking the limit

$$\lim_{x\to 0} x^2\cdot Q(x) = x^2\cdot\frac{1}{x} = 0\checkmark$$

(b) Plugging our ansatz and its derivatives into the ODE gives us

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n} + a_nx^{r+n+1} = 0$$

We notice that the smallest term here is x^r, so we find the coefficients of that and equate it to 0

$$x^r: \qquad r(r-1)a_0 = 0 \Rightarrow r(r-1) = 0$$

This is called the indicial equation, and similar to a characteristic polynomial we solve for the roots to get $r = 0, 1$ and these roots are called the exponents at the singularity. Now we can go ahead and find the coefficients of our general term

$$x^{m+r}: \qquad (r+m)(r+m-1)a_m + a_{m-1} = 0 \Rightarrow a_m = -\frac{a_{m-1}}{(r+m)(r+m-1)}$$

(c) Since the roots differ by an integer multiple we can only solve for the larger one, so

$$r = 1: \qquad a_m = -\frac{a_{m-1}}{m(m+1)}.$$

Now if we look at the first few (or equivalently the last few) terms we can figure out what the pattern will be

$$a_1 = -\frac{a_0}{1\cdot(1+1)} \Rightarrow a_2 = -\frac{a_1}{2\cdot(2+1)} = \frac{a_0}{2\cdot 1\cdot(2+1)\cdot(1+1)} \Rightarrow a_3 = \frac{a_0}{3\cdot 2\cdot 1\cdot(3+1)(2+1)(1+1)}$$

As we continue this we see that

$$a_m = (-1)^n \frac{a_0}{m!(m+1)!} \Rightarrow y_1 = x + \sum_{n=0}^{\infty} (-1)^n \frac{a_0}{m!(m+1)!} x^{n+1}$$

12) It's easy to show the two values are singular points, so lets go to the actual part of the problem. I will solve the problem for $x_0 = 1$. Let $t = x - 1 \Rightarrow x = t + 1$, then the ODE becomes

$$-t(t+2)y'' - (t+1)y' + \alpha^2 y = 0.$$

For these types of problems its easier to keep the equation as is instead of change it into (8), but be careful, our lowest term won't be x^r anymore so keep that in mind. Plugging our ansatz and derivatives into this ODE gives us

$$\sum_{n=0}^{\infty} -(r+n)(r+n-1)a_n t^{r+n} - 2(r+n)(r+n-1)a_n t^{r+n-1} - (r+n)a_n t^{r+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} -(r+n)(r+n-1)a_n t^{r+n} - 2(r+n)(r+n-1)t^{r+n-1} - (r+n)a_n t^{r+n} - (r+n)a_n t^{r+n-1} + \alpha^2 a_n t^{r+n} = 0.$$

We observe that the lowest term is t^{r-1}, so we find the coefficients of this term first,

$$t^{r-1}: \qquad [-2r(r-1) - r]a_0 \Rightarrow -2r^2 + r = r(-2r+1) = 0 \Rightarrow r = 0, \frac{1}{2}.$$

Since these are not integer multiples we can solve for both, but first we need the general term

$$\begin{aligned} t^{r+m}: \qquad & -(r+m)(r+m-1)a_m - 2(r+m+1)(r+m)a_{m+1} - (r+m)a_m - (r+m+1)a_{m+1} + \alpha^2 a_m \\ & = [\alpha^2 - (r+m)^2]a_m - (r+m+1)(2r+2m+1)a_{m+1} = 0 \\ & \Rightarrow a_{m+1} = \frac{\alpha^2 - (r+m)^2}{(r+m-1)(2r+2m+1)} a_m \end{aligned}$$

We could solve for the two solutions, but that gets extremely tedious and long, so on the homework feel free to stop at this point.

Differential equations are hard! With the characteristic polynomial we were able to convert the problem into an algebraic equation, but this only works for simple problems. For harder problems there are special types of transforms called integral transforms.

Definition 3. If $f(t)$ is defined for all $t > 0$ and if $s \in \mathbb{R}$ such that the integral

$$F(S) = \int_0^\infty e^{-st} f(t)dt \tag{9}$$

converges for $s < s_n < \infty$, then $F(s)$ is called the Laplace Transform of $f(t)$ and denoted as $\mathcal{L}\{f(t)\} = F(s)$.

Now lets do some examples

2) We did the sketch in class.

10) Here we use the definition of the Laplace Transform

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}e^{at}\sinh(bt)dt = \int_0^\infty e^{-st}e^{at}\frac{1}{2}(e^{bt} - e^{-bt})dt = \frac{1}{2}\int_0^\infty \left[e^{(b+a-s)t} - e^{(-b+a-s)t}\right] \\
&= \lim_{\tau\to\infty}\frac{1}{2}\int_0^\tau \left[e^{(b+a-s)t} - e^{(-b+a-s)t}\right] = \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{b+a-s}e^{(b+a-s)t} - \frac{1}{-b+a-s}e^{(-b+a-s)t}\right]_0^\tau \\
&= \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{b+a-s}e^{(b+a-s)\tau} - \frac{1}{-b+a-s}e^{(-b+a-s)\tau} - \frac{1}{b+a-s} - \frac{1}{-b+a-s}\right] \\
&= \frac{1}{2}\left[\frac{1}{s-a-b} - \frac{1}{s-a+b}\right] = \frac{b}{(s-a)^2 - b^2}.
\end{aligned}$$

However, notice that we can only take the integral if both $b + a - s < 0$ and $-b + a - s < 0$, which means we need $s - a > |b|$.

14) Again we use the definition

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}e^{at}\cos bt = \int_0^\infty e^{-st}e^{at}\frac{1}{2}(e^{ibt} - e^{-ibt})dt = \frac{1}{2}\int_0^\infty \left(e^{[(a+ib)-s]t} - e^{[(a-ib)-s]t}\right)dt \\
&= \lim_{\tau\to\infty}\frac{1}{2}\int_0^\tau \left(e^{[(a+ib)-s]t} - e^{[(a-ib)-s]t}\right)dt = \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{(a+ib)-s}e^{[(a+ib)-s]t} - \frac{1}{(a-ib)-s}e^{[(a-ib)-s]t}\right]_0^\tau \\
&= \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{(a+ib)-s}e^{[(a+ib)-s]\tau} - \frac{1}{(a-ib)-s}e^{[(a-ib)-s]\tau} - \frac{1}{(a+ib)-s} + \frac{1}{(a-ib)-s}\right] \\
&= \frac{1}{2}\left[\frac{1}{(a-ib)-s} - \frac{1}{(a+ib)-s}\right] = \frac{s-a}{(s-a)^2 + b^2}.
\end{aligned}$$

The condition for this is more difficult, but if we ignore the complex part, which we can do because of certain properties of complex numbers, we get that the integral converges for $s > a$.

22) This one is much easier than the previous two. Notice that after $t = 1$ the function is zero, so our integral becomes

$$F(s) = \int_0^\infty e^{-st}f(t)dt = \int_0^1 te^{-st}dt = -\frac{1}{s^2}e^{-st}(st+1)\Big|_0^1 = -\frac{1}{s^2}e^{-s}(s+1) + \frac{1}{s^2}.$$

We have one more small theoretical consideration. This can be derived using Calc II, but unless you were in my class or Professor Horntrop's you probably didn't see it in Calc II.

Theorem 1. *If $|f(t)| \le g(t)$ for $t \ge M$, $\int_M^\infty g(t)dt$ converges implies $\int_a^\infty f(t)dt$ also converges for $t \ge a$, and if $f(t) \ge g(t)$ for $t \ge M$, $\int_M^\infty g(t)dt$ diverges implies $\int_0^\infty f(t)dt$ diverges for $t \ge a$.*

Lets do one example with this.

28) We have that $|\cos t| \le 1$, so $|e^{-t}\cos t| \le e^{-t}$, and $\int_0^\infty e^{-t}dt = 1$ converges so $\int_0^\infty e^{-t}\cos t dt$ also converges.

6.2 IVPs with Laplace Transforms

I'll "handwave" this section because deeper knowledge is required to properly understand the theory, which exceeds the scope of this course.

In order to apply Laplace transforms to ODEs we have to take the Laplace of the derivatives. Let $Y = \mathcal{L}\{y\}$ and $y' = dy/dt$, then

$$\mathcal{L}\{y'\} = \int_0^\infty e^{-st} y' dt = e^{-st} y\big|_0^\infty + s\int_0^\infty e^{-st} y dt = -y(0) + sY. \tag{10}$$

It should be noted that this integral was done using integration by parts. We can get higher derivatives by induction

$$\mathcal{L}\{y''\} = \mathcal{L}\{(y')'\} = -y'(0) + s\mathcal{L}\{y'\} = -y'(0) - sy(0) + s^2 Y. \tag{11}$$

Lets do a few examples

5) We first recognize what it resembles and try to convert it into that form

$$\frac{2s+2}{s^2+2s+5} = 2\frac{s+1}{(s+1)^2+4} \Rightarrow \mathcal{L}^{-1}\{F(s)\} = 2e^{-t}\cos 2t.$$

8) First we do the partial fractions

$$\frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow As^2 + 4A + Bs^2 + Cs = (A+B)s^2 + Cs + 4A = 8s^2 - 4s + 12.$$

Then we get $A = 3$, $C = -4$, and $B = 5$, then

$$F(s) = \frac{3}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4} \Rightarrow \mathcal{L}\{F(s)\} = 3 + 5\cos 2t - 2\sin 2t.$$

14) We take the Laplace transform of the entire ODE

$$-\cancelto{1}{y'(0)} - s\cancelto{1}{y(0)} + s^2 Y + 4\cancelto{1}{y(0)} - 4sY + 4Y = 0 \Rightarrow (s^2 - 4s + 4)Y = s - 3 \Rightarrow Y = \frac{s-3}{s^2-4s+4}$$

$$\Rightarrow Y = \frac{s-3}{(s-2)^2} = \frac{\cancel{s-2}}{(s-2)^{\cancel{2}}} - \frac{1}{(s-2)^2} \Rightarrow y(t) = e^{2t} - te^{2t}.$$

22) Again we take the Laplace transform of the entire ODE

$$-\cancelto{1}{y'(0)} - s\cancelto{0}{y(0)} + s^2 Y + 2\cancelto{0}{y(0)} - 2sY + 2Y = \frac{1}{s+1} \Rightarrow (s^2 - 2s + 2)Y = \frac{1}{s+1} + 1$$

$$\Rightarrow Y = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2}.$$

The second term is fine, but for the first time we must do partial fractions, which you'll have to use a lot

$$\frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2} \Rightarrow As^2 - 2As + 2A + Bs^2 + Bs + Cs + C = (A+B)s^2 + (B+C-2A)s + 2A + C = 1.$$

Then we get $A = -B \Rightarrow C = 3A$, then $A = 1/5 = -B$, and $C = 3/5$. Hence,

$$Y = \frac{1}{5}\cdot\frac{1}{s+1} + \frac{1}{5}\cdot\frac{-s+8}{s^2-2s+2} = \frac{1}{5}\cdot\frac{1}{s+1} - \frac{1}{5}\cdot\frac{s-1}{(s-1)^2+1} + 7\cdot\frac{1}{(s-1)^2+1}$$

$$\Rightarrow y = \frac{1}{5}[e^{-t} - e^t\cos t + 7e^t \sin t].$$