3.5 Variation of Parameters

Consider the ODE

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

and suppose we have the following characteristic solution

$$y_c = c_1y_1 + c_2y_2 \tag{2}$$

What if for the full solution to (1) we can think of the "constants" as functions; i.e. $y = u_1(x)y_1 + u_2(x)y_2$. We can use this as an ansatz and plug it into the ODE. For the derivative we get

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

We only want one derivative in our final equation so lets force

$$u_1'y_1 + u_2'y_2 = 0 \tag{3}$$

so $y' = u_1y_1' + u_2y_2'$, then

$$y'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''.$$

Plugging this into 1 gives

$$u_1'y_1' + u_2'y_2' + u_1\cancelto{0}{[y_1'' + p(x)y_1' + q(x)y_1]} + u_2\cancelto{0}{[y_2'' + p(x)y_2' + q(x)y_2]} = f(x)$$

Notice that the terms in brackets cancel because they are solutions to the nonhomogeneous ODE. This gives us our second equation

$$u_1'y_1' + u_2'y_2' = f(x) \tag{4}$$

From (3) we get $u_1' = -u_2'y_2/y_1$. We plug this into 4 in order to get an expression for u_2

$$-u_2'y_1'\frac{y_2}{y_1} + u_2'y_2' = f(x) \Rightarrow -u_2'y_1'y_2 + u_2'y_2'y_1 = f(x)y_1 \Rightarrow u_2' = \frac{f(x)y_1}{y_2'y_1 - y_1'y_2} = \frac{f(x)y_1}{W(y_1, y_2)}.$$

Now we plug this into our expression for u_1 to get

$$u_1' = -\frac{f(x)y_2}{W(y_1, y_2)}$$

Then we integrate to get

$$u_1 = -\int \frac{f(x)y_2}{W(y_1, y_2)}dx \tag{5}$$

$$u_2 = \int \frac{f(x)y_1}{W(y_1, y_2)}dx \tag{6}$$

Then plugging back into our original ansatz gives us

$$y = -y_1\int \frac{f(x)y_2}{W(y_1, y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1, y_2)}dx$$

Theorem 1. *Suppose the ODE* (1) *has a unique solution on* I *open. Assume it has the characteristic solution* (2)*. Then*

$$y = -y_1\int \frac{f(x)y_2}{W(y_1, y_2)}dx + y_2\int \frac{f(x)y_1}{W(y_1, y_2)}dx \tag{7}$$

is the general solution.

Now we could just use this theorem for all our problems. The only downfall is that we will have to memorize this formula. So, just in case you forget the formula, do know how to work out the derivation, and try to use the derivation on specific problems.

Ex: $y'' - y' - 2y = 2e^{-t}$.

Solution: $r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow y_c = c_1e^{2t} + c_2e^{-t}$, so $y_1 = e^{2t}$ and $y_2 = e^{-t}$. First we calculate the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -3e^t.$$

Now lets compute our two integrals separately

$$\int \frac{f(t)y_2}{W(y_1, y_2)}dt = \int \frac{e^{-t}\cdot 2e^{-t}}{-3e^t}dt = -\frac{2}{3}\int e^{-3t}dt = -\frac{2}{9}e^{-3t} + c_3.$$

and

$$\int \frac{f(t)y_1}{W(y_1, y_2)}dt = \int \frac{e^{2t}\cdot 2e^{-t}}{-3e^t}dt = \frac{2}{3}\int dt = -\frac{2}{3}t + c_4.$$

Then plugging this back into (7) gives

$$y = -e^{2t}\left[-\frac{2}{9}e^{-3t} + c_3\right] + e^{-t}\left[\frac{2}{3}t + c_4\right] = \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t} - c_3e^{2t} + c_4e^{-t} = c_5e^{-t} - \frac{2}{3}te^{-t} - c_3e^{2t}.$$

Ex: $y'' - 2y' + y = e^t/(1+t^2)$.

Solution: $r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow y_c = c_1e^t + c_2te^t$, then $y_1 = e^t$ and $y_2 = te^t$, then we compute the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}.$$

Now we compute the two integrals

$$\int \frac{f(t)y_2}{W(y_1,y_2)}dt = \int \frac{te^t \cdot e^t/(1+t^2)}{e^{2t}}dt = \int \frac{tdt}{1+t^2} = \frac{1}{2}\ln(1+t^2) + c_3.$$

and

$$\int \frac{f(t)y_1}{W(y_1,y_2)}dt = \int \frac{e^t \cdot e^t/(1+t^2)}{e^{2t}}dt = \int \frac{dt}{1+t^2} = \tan^{-1} t + c_4$$

Plugging this into (7) gives

$$y = -\frac{1}{2}e^t\ln(1+t^2) + te^t\tan^{-1} t - c_3e^t + c_4te^t.$$

Ex: $y'' - 5y' + 6y = g(t)$.

Solution: $r^2 - 5r + 6 = (r-3)(r-2) = 0 \Rightarrow y_c = c_1e^{3t} + c_2e^{2t}$, so $y_1 = e^{3t}$ and $y_2 = e^{2t}$. The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = -e^{5t}$$

Plugging this into (7) gives us

$$y = -e^{3t}\int \frac{e^{2t}g(t)}{-e^{5t}}dt + e^{2t}\int \frac{e^{3t}g(t)}{-e^{5t}}dt = e^{3t}\int e^{-3t}g(t)dt - e^{2t}\int e^{-2t}g(t)dt$$

Ex: $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$, $t > 0$; $y_1(t) = t$, $y_2(t) = te^t$.

Solution: We must first convert this into standard form

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t.$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t.$$

Then we plug into (7) to get

$$y = -t\int \frac{te^t \cdot 2t}{t^2e^t}dt + te^t\int \frac{t \cdot 2t}{t^2e^t}dt = -t\int 2dt + te^t\int 2e^{-t}dt = -2t^2 + c_1tc_2te^t.$$

So the particular solution is

$$y_p = -2t^2$$

Ex: $x^2y'' + xy' + (x^2 - 0.25)y = g(x)$, $x > 0$; $y_1(x) = x^{-1/2}\sin x$, $y_2(x) = x^{-1/2}\cos x$.

Solution: We convert this to standard form

$$y'' + \frac{1}{x}y' + \frac{x^2 - 0.25}{x^2}y = \frac{g(x)}{x^2}.$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} x^{-1/2}\sin x & x^{-1/2}\cos x \\ -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x & -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x \end{vmatrix} = -\frac{1}{x}$$

Then plugging into (7) gives

$$y = -x^{-1/2}\sin x\int \frac{x^{-1/2}\cos x \cdot g(x)/x^2}{-1/x}dx + x^{-1/2}\cos x\int \frac{x^{-1/2}\sin x \cdot g(x)/x^2}{-1/x}dx$$

$$= x^{-1/2}\sin x\int \frac{\cos x g(x)}{x\sqrt{x}}dx - x^{-1/2}\cos x\int \frac{\sin x g(x)}{x\sqrt{x}}dx.$$