1.1 Classification of Differential Equations

We discussed the difference between ODEs and PDEs in class. In this course we only concentrate on ODEs.

Definition 1. The order of an ODE is the order of the highest derivative.

In the previous section we dealt with all first order ODEs. I gave some examples of higher order ODEs in class, and there are plenty of examples for that in the book.

Definition 2. Consider the ODE

$$F(t, y, y', \dots, y^{(n)}) = 0, \tag{1}$$

then the ODE is said to be linear if F is a linear function with respect to $y, y', \dots, y^{(n)}$.

Definition 3. We say an ODE is nonlinear if it is not linear.

Again we did examples of these in class and there are many examples in the book.

Definition 4. A function $y(t)$ is said to be a solution on (a, b) if for every $t \in (a, b)$, $y(t), y'(t), \dots, y^{(n)}(t)$ exists and satisfies $F(t, y, y', \dots, y^{(n)}) = 0$.

Notice, a solution need not be unique. We discussed examples of this in class.

For the next few examples we will verify a certain function is a solution to the given ODE. For these problems you want to first check the existence of the derivatives and then plug into the ODE to verify the ODE is satisfied.

Ex: $y'' - y = 0$; a) $y_1(t) = e^t$, b) $y_2(t) = \cosh t$

Solution:

a) $y_1' = y_1'' = e^t$ and $y_1'' - y_1 = e^t - e^t = 0$.

b) $y_2' = \sinh t$ and $y_2'' = \cosh t$, furthermore $y_2'' - y_2 = \cosh t - \cosh t = 0$.

Ex: $ty' - y = t^2$; $y = 3t + t^2$

Solution: $y' = 3 + 2t$ and $ty' - y = (3t + 2t^2) - (3t + t^2) = t^2$.

Ex: $y'''' + 4y''' + 3y = t$; a) $y_1 = t/3$, b) $y_2 = e^{-t} + t/3$

Solution:

a) $y_1' = 1/3$, $y_1'' = y_1''' = y_1^{(4)} = 0$ and $y^{(4)} + 4y''' + 3y = 3 \cdot t/3 = t$.

b) $y_2' = -e^{-t} + 1/3$, $y_2'' = y^{(4)} = -y''' = e^{-t}$ and $y^{(4)} + 4y''' + 3y = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$.

2.2 Separable Equations

Separable equations are the easiest equations to solve. This why it's extremely important to recognize separable equations. It will save you a lot of work! One thing we will notice right away is that Autonomous first order ODEs are always separable.

Definition 5. An ODE is separable if it can be written in the form $f(x)dx = g(y)dy$.

For the next few problems we will solve some separable equations.

Ex: $y' = x^2/y$

Solution: We separate the equation by "moving" y to the left and dx to the right,

$$ydy = x^2dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{3}x^3 + C_0 \Rightarrow y = \pm\sqrt{\frac{2}{3}x^3 + C_1};\ y \neq 0.$$

Ex: $y' = \dfrac{3x^2 - 1}{3 + 2y}$

Solution: We separate the equation by moving $3 + 2y$ to the left and dx to the right,

$$(3 + 2y)dy = (3x^2 - 1)dx \Rightarrow 3y + y^2 = x^3 - x + C;\ y \neq -\frac{3}{2}.$$

Ex: $y' = (1 - 2x)/y$

Solution: This type of problem is called an "Initial Value Problem" (IVP). The idea is to use the Initial Value to solve for the constant of integration. First we separate the problem by moving y to the left and dx to the right,

$$ydy = (1 - 2x)dx \Rightarrow \frac{1}{2}y^2 = x - x^2 + C.$$

Now, the initial value tells us that $y = -2$ when $x = 1$, so if we plug this into the above equation we get that $C = 2$, so plugging it back in and solving for y gives,

$$y = -\sqrt{2x - 2x^2 + 4};\ y \neq 0.$$

Notice we only chose the negative branch of the root because the initial condition starts with negative for the y value and we know that $y \neq 0$ so the solution can't magically cross into the positive branch, so we must stay on the negative branch for all time.

For part b and c, we did the plot in class, and the domain of existence is $-1 < x < 2$.

Ex: $\dfrac{dy}{dt} = ty\dfrac{4-y}{1+t}$

Solution: Separating gives us,

$$\int \frac{dy}{y(4-y)} = \int \frac{tdt}{1+t} \Rightarrow \frac{1}{4}\int\left(\frac{1}{y} + \frac{1}{4-y}\right)dy = \int \frac{u-1}{u}du \Rightarrow \frac{1}{4}\left[\ln|y| - \ln|4-y|\right] = u - \ln|u| + C_0$$

$$\Rightarrow \ln|\frac{y}{4-y}| = 4t - 4\ln|1+t| + C_1 \Rightarrow \frac{y}{4-y} = e^{4t}\frac{K}{(1+t)^4};$$

plugging in the initial condition gives us $K = y_0/(4-y_0)$, then the full solution is

$$y = \frac{\frac{4e^{4t}y_0}{(1+t)^4(4-y_0)}}{1 + \frac{e^{4t}y_0}{(1+t)^4(4-y_0)}} \tag{2}$$

(a) As $t \to \infty$, $y \to 4$.

(b) $y_0 = 2 \Rightarrow K = 1$, so when $y = 3.99$, $T \approx 2.84367$ (via wolfram).

(c) Now, if $t = 2$, $y/(4-y) = e^8 e^{-4\ln 3}K$, then $y = 3.99 \Rightarrow y_0 \approx 3.6622$ and $y = 4.1 \Rightarrow y_0 \approx 4.4042$. This gives us an interval of $3.6622 \leq y_0 \leq 4.4042$.