2.3 - 2.4 Linear Equations and Method of Integrating Factors

Consider the ODE $\frac{dy}{dx} - \frac{y}{x} + \frac{f(x)}{x} = 0$. This is clearly not separable.

Now consider the ODE $t^2\frac{dx}{dt} + 2xt = t$. This too is not separable, but we can make it separable by employing a small trick. Notice that $t^2\frac{dx}{dt} + 2xt = \frac{d}{dt}(xt^2)$, so the ODE becomes, $\frac{d}{dt}(xt^2) = t$, which is separable. This is what is referred to as an "exact ODE". So we get,

$$\frac{d}{dt}(xt^2) = t \Rightarrow d(xt^2) = tdt \Rightarrow \int d(xt^2) = \int tdt \Rightarrow xt^2 = \frac{1}{2}t^2 + C \Rightarrow x = \frac{1}{2} + Ct^{-2}.$$

This is the idea. If we encounter an equation that isn't separable we need to change it in some way that makes it separable. Lets look at the first equation again and write it in differential form, i.e.

$$\frac{dy}{dx} - \frac{y}{x} + \frac{f(x)}{x} = 0 \Rightarrow xdy - ydx + f(x)dx = 0.$$

Notice, that $xdy - ydx$ is almost quotient rule, we just need to finish the denominator, which we notice should be x^2, so let's multiply through by $1/x^2$,

$$\frac{xdy - ydx}{x^2} + \frac{f(x)}{x^2}dx = 0 \Rightarrow d\left(\frac{y}{x}\right) = -\frac{f(x)}{x^2}dx \Rightarrow \int d\left(\frac{y}{x}\right) = -\int \frac{f(x)}{x^2}dx$$

$$\Rightarrow \frac{y}{x} = -\int \frac{f(x)}{x^2}dx \Rightarrow y = -x\int \frac{f(x)}{x^2}dx.$$

This is called the method of "integrating factors", where $1/x^2$ is called the "integrating factor", which are delineated in the following definition.

Definition 1. Consider an ODE of the form

$$\frac{dy}{dx} + p(x)y = g(x). \tag{1}$$

We call $\mu(x)$ an <u>integrating factor</u> if

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right]$$

is an exact ODE, i.e.

$$\mu(x)\left[\frac{dy}{dx} + p(x)y = g(x)\right] \Leftrightarrow d(\mu(x)y) = \mu(x)g(x)dx. \tag{2}$$

All we need to do now is figure out what $\mu(x)$ is in general, but fortunately Leibniz already did that for us,

$$\mu(x) = \exp\left(\int^x p(\xi)d\xi\right). \tag{3}$$

In the following examples we use the method of integrating factors to solve our ODE,

Ex: $y' - 2y = 3e^t$

   Solution: The integrating factor is $\mu = \exp\left(\int^t -2ds\right) = e^{-2t}$. Now, we use our method to get,

$$e^{-2t}y = 3\int e^{-t}dt = -3\int e^{-t}dt = -3e^{-t} + C \Rightarrow y = -3e^t + Ce^{2t}.$$

Now, notice if $C > 0$, $y \to \infty$ as $t \to \infty$, and if $C \le 0$, $y \to -\infty$ as $t \to \infty$.

Ex: $ty' - y = t^2e^{-t}$; $t > 0$

   Solution: The integrating factor is $\mu = \exp\left(\int^t -1/sds\right) = 1/t$, then

$$\frac{y}{t} = \int e^{-t}dt = -e^{-t} + C \Rightarrow y = te^{-t} + Ct.$$

Now, notice if $C = 0$, $y \to 0$ as $t \to \infty$, and if $C \ne 0$, $y \to \infty$ as $t \to \infty$.

Ex: $y' - \frac{1}{2}y = 2\cos t;\ y(0) = a$

Solution: The integrating factor is $\mu = \exp\left(\int^t -(1/2)ds\right) = e^{-t/2}$. Then,

$$e^{-t/2}y = 2\int e^{-t/2}\cos t\,dt = \frac{4}{5}e^{-t/2}(2\sin t - \cos t) + C.$$

We did the integration in class. Know how to do the integration! Then, we get

$$y = \frac{4}{5}(2\sin t - \cos t) + Ce^{t/2}.$$

From the initial condition we get $C = a + 4/5$. We see that the behavior of the system changes at $C = 0$, so $a_0 = -4/5$. Now, when $a = -4/5$, y is oscillatory as $t \to 0$, specifically $y \to \frac{4}{5}(2\sin t - \cos t)$. Furthermore, if $a < -4/5$, $y \to -\infty$, and if $a > -4/5$, $y \to \infty$.

30. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

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Solution: The integrating factor is $\mu = \exp\left(-\int^t ds\right) = e^{-t}$. Then we get

$$e^{-t}y = \int \left(e^{-t} + 3e^{-t}\sin t\right)dt = -e^{-t} + 3\int e^{-t}\sin t\,dt.$$

We employ integration by parts for $\int e^{-t}\sin t\,dt$ with $u = e^{-t} \Rightarrow du = -e^{-t}dt$ and $dv = \sin t\,dt \Rightarrow v = -\cos t$,

$$\int e^{-t}\sin t\,dt = -e^{-t}\cos t + \int e^{-t}\cos t\,dt$$

We employ by parts again with $u = e^{-t} \Rightarrow du = -e^{-t}dt$ and $dv = \cos t\,dt \Rightarrow v = \sin t$,

$$\int e^{-t}\sin t\,dt = -e^{-t}\cos t + e^{-t}\sin t - \int e^{-t}\sin t\,dt \Rightarrow \int e^{-t}\sin t\,dt = -\frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t.$$

Plugging this back into our ODE gives,

$$e^{-t}y = -e^{-t} - \frac{3}{2}e^{-t}\cos t + \frac{3}{2}e^{-t}\sin t + C \Rightarrow y = -1 - \frac{3}{2}\cos t + \frac{3}{2}\sin t + Ce^{t}.$$

The initial condition gives us $y_0 = -1 - 3/2 + C = -5/2 + C$, then our solution is

$$\Rightarrow y = -1 - \frac{3}{2}\cos t + \frac{3}{2}\sin t + (y_0 + 5/2)e^{t}. \tag{4}$$

Then for $y > -5/2$, $y \to \infty$ as $t \to \infty$. For $y < -5/2$, $y \to -\infty$ as $t \to \infty$. However, for $y_0 = -5/2$, y oscillates, but remains finite.