

3.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

It should be noted that while this chapter is on second order ODEs, we will develop the theory for higher order ODEs because the theory is exactly the same! Let us first go over some definitions we might not know,

Definition 1. An ODE is homogeneous if it is of the form

$$p_n(t)y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \dots + p_2(t)y''(t) + p_1(t)y'(t) + p_0(t)y(t) = 0. \tag{1}$$

So an example of a second order homogeneous ODE would be $p_2y'' + p_1y' + p_0y = 0$.

Definition 2. An ODE is said to be nonhomogeneous if it's not homogeneous.

An example of a second order nonhomogeneous ODE would be $p_2y'' + p_1y' + p_0y = f(t)$. In this section we will only deal with constant coefficients which mean each $p_n(t) = a_n$ where $a_0, a_1, \dots, a_{n-1}, a_n$ are all constants.

Now, we consider a special case of Eq. (1): $y' + ay = 0$ We know how to solve this, we simply use separation to get $y = ke^{-ax}$. So, we can “guess” that the form of the solutions for Eq. (1) with constant coefficients will be $y = ke^{rx}$. Now, we plug this guess in to see what the solutions exactly are. Notice that the nth derivative is, $y^{(n)} = kr^n e^{rx}$, so plugging this into (1) with $p_n(t) = a_n$ gives,

$$\begin{aligned} a_nkr^n e^{rx} + a_{n-1}kr^{n-1}e^{rx} + \dots + a_2kr^2e^{rx} + a_1kre^{rx} + a_0ke^{rx} &= 0 \\ \Rightarrow ke^{rx} (a_nr^n + a_{n-1}r^{n-1} + \dots + a_2r^2 + a_1r + a_0) &= 0. \end{aligned}$$

Now, all we have to do is solve the polynomial equation. Since this is an nth order polynomial, there will be n solutions, i.e. $r = r_1, r_2, \dots, r_{n-1}, r_n$. Since the polynomial equation has n solutions, the ODE will also have n solutions, so by superposition we get,

$$y = k_1e^{r_1x} + k_2e^{r_2x} + \dots + k_{n-1}e^{r_{n-1}x} + k_n e^{r_nx}.$$

We have just proved a theorem,

Theorem 1. Consider the ODE

$$a_ny^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \dots + a_2y''(x) + a_1y'(x) + a_0y(x) = 0. \tag{2}$$

such that $a_0, a_1, \dots, a_{n-1}, a_n$ are constants. Then,

$$y = k_1e^{r_1x} + k_2e^{r_2x} + \dots + k_{n-1}e^{r_{n-1}x} + k_n e^{r_nx}, \tag{3}$$

where $k_1, k_2, \dots, k_{n-1}, k_n$ are constants and $r_1, r_2, \dots, r_{n-1}, r_n$ satisfy the polynomial equation

$$a_nr^n + a_{n-1}r^{n-1} + \dots + a_2r^2 + a_1r + a_0 = 0, \tag{4}$$

only if $r_1 \neq r_2 \neq \dots \neq r_{n-1} \neq r_n$.

Definition 3. We call Eq. (4) the characteristic equation of ODE (2), and the polynomial is called the characteristic polynomial.

Now, lets do a few problems,

Ex: $y'' + 2y' - 3y = 0$

Solution: The characteristic polynomial is $r^2 + 2r - 3$, so

$$r^2 + 2r - 3 = 0 \Rightarrow (r + 3)(r - 1) = 0 \Rightarrow r = 1, -3 \Rightarrow y = c_2e^x + c_2e^{-3x}.$$

Ex: $y'' - 9y' + 9 = 0$

Solution: The characteristic polynomial is $r^2 - 9r + 9$, so

$$r = \frac{1}{2}(9 \pm 3\sqrt{5}) \Rightarrow y = c_1e^{\frac{1}{2}(9+3\sqrt{5})x} + c_2e^{\frac{1}{2}(9-3\sqrt{5})x}.$$

Ex: $y'' + 3y' = 0; y(0) = -2, y'(0) = 3$

Solution: The characteristic polynomial is $r^2 + 3r$, so

$$r = 0, -3 \Rightarrow y = c_1 + c_2e^{-3x},$$

and from the initial conditions we get $y = -1 - e^{-3x}$.

Ex: Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.

Solution: Here they give us the solution and we have to extract the ODE. Notice that from the solution we deduce

$$r = -\frac{1}{2}, -2 \Rightarrow (r + \frac{1}{2})(r + 2) = r^2 + \frac{5}{2}r + 1 = 0 \Rightarrow y'' + \frac{5}{2}y' + y = 0.$$

Ex: Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \rightarrow \infty$.

Solution: This is kind of a silly question, but since there is a similar one on the homework lets do it. We solve the ODE as per usual,

$$r^2 - r - 2 = (r - 2)(r + 1) = 0 \Rightarrow r = -1, 2 \Rightarrow y = c_1 e^{-x} + c_2 e^{2x}.$$

From the initial condition we have the equations $c_1 + c_2 = \alpha$ and $2c_2 - c_1 = 2$, so $3c_2 = \alpha + 2$. This means that if $\alpha = -2$, as $t \rightarrow \infty$, $y \rightarrow 0$. However, for the second part of the problem there are no solutions that always blow up because we have a negative exponential term that will persist.

In each of Problems 23 and 24, determine the values of α , if any, for which all solutions tend to zero as $t \rightarrow \infty$; also determine the values of α , if any, for which all (nonzero) solutions become unbounded as $t \rightarrow \infty$.

23. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$

24. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$

24) For this problem the ODE itself has the parameter α . This leads to interesting conclusions without even solving, but the easiest most intuitive way to come to those conclusions will be by solving, even though it is more tedious and time consuming. We solve the ODE,

$$r^2 + (3 - \alpha)r - 2(\alpha - 1) = 0 \Rightarrow (r - (\alpha - 1))(r + 2) = 0 \Rightarrow r = -2, \alpha - 1 \Rightarrow y = c_2 e^{-2x} + c_2 e^{(\alpha-1)x}.$$

So, for $\alpha < 1$, $y \rightarrow \infty$. If $\alpha = 1$, $y \rightarrow c_2$, and if $\alpha > 1$, and $y \rightarrow \pm\infty$ only if $c_2 \neq 0$.