3.1 Theory of linear equations

Last time we discussed ODEs of the form,

$$p_n(x)y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \cdots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = 0.$$

Now lets look at the general case of,

$$p_n(x)y^{(n)}(x) + p_{n-1}y^{(n-1)}(x) + \cdots + p_2(x)y''(x) + p_1(x)y'(x) + p_0(x)y(x) = g(x).$$

Lets put this in standard form by dividing through by $p_n(x)$ and naming the new functions "q" and "f",

$$y^{(n)}(x) + q_{n-1}(x)y^{(n-1)}(x) + \cdots + q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = f(x). \tag{1}$$

Consider the simple ODE,

$$y' + q(x)y = f(x);\; q(x) = \begin{cases} 1 \text{ if } x \text{ is irrational,} \\ 0 \text{ if } x \text{ is rational;} \end{cases}$$

In order to solve this we would need to use integrating factors, however notice that q is not integrable (in the usual fashion), so we can't solve this - in fact it has no unique solution. So, we need conditions on $q's$ and f to guarantee that we can find a unique solution. We outline this in the next theorem, however one should proceed with caution because this only works for linear ODEs.

Theorem 1. *Consider ODE (1) with initial conditions: $y(x_0) = a_0$, $y'(x_0) = a_1, \ldots, y^{(n-1)}(x_0) = a_{n-1}$.*
Then, if $q_{n-1}, q_{n-2}, \ldots, q_2, q_1, q_0$ are continuous on a common interval I containing x_0, the IVP has exactly one solution on I.

Now we proceed to defining certain important ideas that we will use in our following theorems.

Definition 1. The set of functions $\{h_1, h_2, \ldots, h_{n-1}, h_n\}$ are said to be linearly independent if $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n \neq 0$, otherwise it is said to be linearly dependent.

Definition 2. The expression $c_1h_1 + c_2h_2 + \cdots + c_{n-1}h_{n-1} + c_nh_n$ is said to be a linear combination of $h_1, h_2, \ldots, h_{n-1}, h_n$.

Last time we talked about superposition. We will pose it more rigorously in the next theorem. First consider the homogeneous ODE in standard form,

$$y^{(n)}(x) + q_{n-1}(x)y^{(n-1)}(x) + \cdots + q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = 0. \tag{2}$$

Theorem 2. *If $y_1, y_2, \ldots, y_{n-1}, y_n$ are solutions to (2), then any linear combination of y's are also solutions.*

For example, $y = c_1y_1 + c_2y_2$, $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, etc. are also solutions.
Now we define what the Wronskian is, which will be a major part of this section.

Definition 3. Suppose $h_1(x), h_2(x), \ldots, h_{n-1}, h_n$ are functions with $n-1$ derivatives, then the Wronskian is defined to be the following determinant,

$$W = \begin{vmatrix} h_1 & h_2 & \cdots & h_n \\ h_1' & h_2' & \cdots & h_n' \\ h_1'' & h_2'' & \cdots & h_n'' \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(n-1)} & h_2^{(n-1)} & \cdots & h_n^{(n-1)} \end{vmatrix} \tag{3}$$

Theorem 3. *Suppose $y_1, y_2, \ldots, y_{n-1}, y_n$ are solutions to (2) on I, with the usual initial conditions, then $W \neq 0$ guarantees they are linearly independent on $\in I$.*

So the rewording of the above theorem implies that if the Wronskian is zero at a single point then the function may still be linearly independent.
The next definition and theorem will allow us to find guaranteed linearly independent solutions, but note that these are not necessarily the only linearly independent solutions.

Definition 4. The set of all linearly independent solutions of an ODE is called the fundamental set of that ODE.

For the remaining theorems consider the second order ODE,

$$y'' + q_1(x)y' + q_0(x)y = 0. \tag{4}$$

Theorem 4. *Consider ODE (4), and let y_1, y_2 solve (4) for $x \in I$ such that $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x_0) = 0$, $y_2'(x_0) = 1$. Then, y_1, y_2 form a fundamental set of (4).*

The following theorem is a theorem we use in section 3.3.

Theorem 5. *If $y = u(t) + iv(t)$ solves (4) on I, then so does u and v independently, i.e. if $y = c_1u + ic_2v$ is a solution, so is $y = c_3u + c_4v$.*

The next theorem gives us a formula to compute the Wronskian without having to take a determinant, but it only works for second order ODEs.

Theorem 6 (Abel)**.** *The Wronskian of y_1, y_2 for (4) can be written as,*

$$W(y_1, y_2) = c\exp\left(-\int q_1(x)dx\right), \tag{5}$$

and is zero (if $c = 0$) or nonzero (if $c \neq 0$) for all $x \in I$.

Now lets do some example problems,

Ex: Find the Wronskian of e^{2t} and $e^{-3t/2}$.
Solution: The derivatives are $2e^{2t}$ and $(-3/2)e^{-3t/2}$, so our Wronskian is,

$$W = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{3}{2}e^{2t-3t/2} - 2e^{2t-3t/2}.$$

Ex: Find the Wronskian of e^{-2t} and te^{-2t}
Solution: The derivatives are $-2e^{-2t}$ and $e^{-2t} - 2te^{-2t}$, so our Wronskian is,

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}.$$

Ex: Find the longest interval for guaranteed existence and uniqueness of $t(t-4)y'' + 3ty' + 4y = 2$; $y(3) = 0$, $y'(3) = -1$.
Solution: We put the ODE in standard form,

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}.$$

Notice, that this has discontinuities at $t = 0, 4$, and since we need to include the initial condition, the largest domain where a unique solution exists is $t \in (0, 4)$.

Ex: Find the longest interval for guaranteed existence and uniqueness of $(x-3)y'' + xy' + (\ln|x|)y = 0$; $y(1) = 0$, $y'(1) = 1$.
Solution: Again we convert the ODE into standard form,

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0$$

This is discontinuous when $x = 0, 3$, so our largest domain where a unique solution containing the initial condition exists is $x \in (0, 3)$.

Ex: If the Wronskian of f and g is $3e^{4t}$ and $f(t) = e^{2t}$, what is $g(t)$?
Solution: Here we have an inverse problem. We need to find a g that satisfies the Wronskian given, so lets take the Wronskian and see what we get,

$$W = \begin{vmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{vmatrix} = e^{2t}g' - 2e^{2t}g = e^{2t}(g' - 2g) = 3e^{4t} \Rightarrow g' - 2g = 3e^{2t}$$

So we have to solve this first order ODE via integrating factor,

$$\mu = \exp\left(-\int^t 2d\tau\right) \Rightarrow \int d(e^{-2t}g) = \int 3dt \Rightarrow e^{-2t}g = 3t + C \Rightarrow g = 3te^{2t} + Ce^{2t}.$$

Ex: Find the fundamental set of solutions of $y'' + 4y' + 3y = 0$; $t_0 = 1$ using the theorem in this section.
Solution: We go straight to the characteristic polynomial, $r^2+4r+3 = (r+1)(r+3) = 0 \Rightarrow r = -1, -3$, so our general solution is $y = c_1e^{-x} + c_2e^{-3x}$. Now, by Theorem 4, we solve two different IVPs for this ODE: $y_1(1) = c_1e^{-1} + c_2e^{-3} = 1$ and $y_1'(1) = -c_1e^{-1} - 3c_2e^{-3} = 0$. By summing the two equations we get $-2c_2e^{-3} = 1 \Rightarrow c_2 = -e^3/2$, this gives $c_1 = 3e/2$, so our first solution is $y_1 = \frac{3}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}$. For the second solution we have $y_2(1) = c_1e^{-1} + c_2e^{-3} = 0$ and $y_2'(1) = -c_1e^{-1} - 3c_2e^{-3}$. We easily get $c_2 = -e^3/2$ and then $c_1 = e/2$, which gives us a solution of $y_2 = \frac{1}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}$. So, the following equations make a fundamental set of the ODE,

$$y_1 = \frac{3}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}; \; y_2 = \frac{1}{2}e^{(1-x)} - \frac{1}{2}e^{3(1-x)}.$$

Ex: Does $y_1 = x$ and $y_2 = \sin x$ constitute a fundamental set of solutions to $(1 - x\cot x)y'' - xy' + y = 0$; $0 < x < \pi$?

Solution: For the first solution we have $y_1' = 1 \Rightarrow y_1'' = 0 \Rightarrow -xy_1' + y_1' = 0$. For the second solution we have $y_2' = \cos x \Rightarrow y_2'' = -\sin x$, then $(1 - x\cot x)(-\sin x) - x\cos x + \sin x = -\sin x + x\cos x - x\cos x + \sin x = 0$. Now, we take the Wronskian of these,

$$W = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0 \text{ for } x \in (0, \pi).$$

So, they are linearly independent on that domain.

Ex: If W is the Wronskian of f and g, and if $u = 2f - g$ and $v = f + 2g$, what is the Wronskian of u and v?

Recall the Wronskians in 2D is $W(f, g) = fg' - f'g$ and $W(u, v) = uv' - u'v$. Then

$$\begin{aligned} W(u, v) &= (2f - g)(f' + 2g') - (2f' - g')(f + 2g) = 2ff' + 4fg' - gf' - 2gg' - 2f'f - 4f'g + g'f + 2g'g \\ &= 4[fg' - f'g] + [fg' - f'g] = 5W(f, g). \end{aligned}$$