3.3 Homogeneous Linear Equations with Constant Coefficients

Complex Conjugate Roots: Again consider the ODE,

$$ay'' + by' + cy = 0, \tag{1}$$

which has the characteristic polynomial equation,

$$ar^2 + br + c = 0. \tag{2}$$

Using the quadratic formula we get,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What if $b^2 - 4ac < 0$? Then r is of the form $r = \xi \pm i\theta$ where $\xi, \theta \in \mathbb{R}$, but this means r is a complex conjugate. However, we do the same thing as usual to get,

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(\xi + i\theta)x} + c_2 e^{(\xi - i\theta)x} = e^{\xi x}\left(c_1 e^{i\theta x} + c_2 e^{-i\theta x}\right).$$

We need to deal with the part inside the parentheses, and we do this by what's called, *Euler's Identity*. And we can derive this fairly easily by using Taylor series, since we know the taylor series,

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+1}}{(2n+1)!} = \cos t + i \sin t. \tag{3}$$

Then our solution becomes,

$$y = e^{\xi x}[c_1(\cos\theta x + i\sin\theta x) + c_2(\cos\theta x - i\sin\theta x)] = e^{\xi x}[(c_1 + c_2)\cos\theta x + i(c_1 - c_2)\sin\theta x].$$

However, we only want real solutions. Notice that $\cos\theta x$ and $\sin\theta x$, with the proper constant coefficients, are solutions to (1) independently. So, by the theorem from the previous section, $y = e^{\xi x}(A\cos\theta x + B\sin\theta x)$ is also a solution. We have just developed a theorem,

Theorem 1. *If* (2) *has complex roots, i.e.* $r = \xi + i\theta$*, then the general solution of* (1) *is,*

$$y = e^{\xi x}(A\cos\theta x + B\sin\theta x). \tag{4}$$

Now, lets do some examples,

Ex: Applying Euler's identity, $e^{2 - i\pi/2} = e^2(\cos\pi/2 - i\sin\pi/2) = -ie^2$.

Ex: $\frac{1}{\pi}e^{i2\ln\pi} = \frac{1}{\pi}(\cos(2\ln\pi) + i\sin(2\ln\pi))$.

Ex: $y'' + 2y' + 2y = 0$.
Solution: We go to the characteristic polynomial, $r^2 + 2r + 2 = 0$ and use the quadratic formula, $r = (-2 \pm \sqrt{4-8})/2 = -1 \pm i$, which gives us a general solution of $y = e^{-t}(A\cos t + B\sin t)$.

Ex: $y'' + 4y' + 5y = 0$; $y(0) = 1$, $y'(0) = 0$.
Solution: Again our characteristic polynomial gives, $r^2 + 4r + 5 = 0$, and the quadratic formula gives, $r = (-4 \pm \sqrt{-4})/2 = -2 \pm i$, so our general solution is $y = e^{-2t}(A\cos t + B\sin t)$. Now we go to our initial conditions: $y(0) = A = 1$. Then, $y'(t) = -2e^{-2t}(\cos t + B\sin t) + e^{-2t}(-\sin t + B\cos t)$, so $y'(0) = -2 + B = 0 \Rightarrow B = 2$. So, our solution is $y = e^{-2t}(\cos t + 2\sin t)$.

Ex: $y'' + y = 0$; $y(\pi/3) = 2$, $y'(\pi/3) = -4$.
Solution: As per usual we have $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y = A\cos t + B\sin t$. From the first initial condition we have, $y(\pi/3) = A/2 + \sqrt{3}B/2 = 2 \Rightarrow A = 4 - \sqrt{3}B$. From the second initial condition we have, $y'(\pi/3) = -\sqrt{3}A/2 + B/2 = -4 \Rightarrow -\sqrt{3} + 3B/2 + B/2 = -2\sqrt{3} + 2B = -4 \Rightarrow B = \sqrt{3} - 2$. So, we get $A = 1 + 2\sqrt{3}$. Then our solution is $y = (1 + 2\sqrt{3})\cos t + (\sqrt{3} - 2)\sin t$.

Repeated Roots: Again consider a second order homogeneous IVP with it's respective characteristic polynomial equation,

$$y'' + by' + cy = 0;\ y(0) = A, y'(0) = B \tag{5}$$

$$r^2 + br + c = 0 \tag{6}$$

Then our roots (also called eigenvalues) are $r = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$. What if $b^2 - 4c = 0 \Rightarrow c = b^2/4$? Then $r_{1,2} = -b/2$. If we plug this in as usual we get $y = c_1 e^{-bx/2} + c_2 e^{-bx/2} = (c_1 + c_2)e^{-bx/2}$. However, this only gives us one constant so there is no way we can satisfy the two initial conditions. So, we need another solution in addition to the one we have.

Suppose the "constant" $c_1 + c_2$ is not a constant, but rather a function of x; i.e. $y = v(x)e^{-bx/2}$. We have to figure out if a v will satisfy our ODE, and if so, what v is it. We want to plug into 5. The derivatives are

$$y' = v'(x)e^{-bx/2} - \frac{b}{2}e^{-bx/2}v(x) \Rightarrow y'' = v''(x)e^{-bx/2} - be^{-bx/2}v'(x) + \frac{b^2}{4}e^{-bx/2}v(x).$$

Plugging into the ODE gives

$$e^{-bx/2}\left(v'' + (-b+b)v' + \left(\frac{b^2}{4} - \frac{b^2}{2} + \frac{b^2}{4}\right)\right) = e^{-bx/2}v'' = 0$$

Since $\exp(-bx/2)$ can't be zero in finite x, $v'' = 0 \Rightarrow v' = c_3 \Rightarrow v = c_3 x + c_4$, which gives us a solution of

$$y = (c_3 x + c_4)e^{-bx/2}.$$

We still don't know if this is a legitimate solution or not yet, but let's write down the theorem anyway and then prove it.

Theorem 2. *Consider the ODE*

$$ay'' + by' + c = 0. \tag{7}$$

If the characteristic polynomial has repeated roots; i.e. $r_{1,2} = \lambda$, *then the general solution to 7 is,*

$$y = (c_1 + c_2 x)e^{\lambda x}. \tag{8}$$

Proof. Clearly 8 is a solution to 7, which we verified by differentiating and plugging into the ODE. Furthermore, $W(e^{\lambda x}, xe^{\lambda x}) = e^{2\lambda x} \neq 0$, which we calculated in class. □

Now, lets solve some problems before moving onto the second part of this section.

Ex: $9y'' + 6y' + y = 0$.

Solution: The characteristic equation is $9r^2 + 6r + 1 = 0 \Rightarrow r = -1/3$, then our solution is $y = (c_1 + c_2 x)e^{-x/3}$.

Ex: $16y'' + 24y' + 9y = 0$.

Solution: As per usual, $16r^2 + 24r + 9 = 0 \Rightarrow r = -3/4 \Rightarrow y = (c_1 + c_2 x)e^{-3x/4}$.

Ex: $4y'' + 12y' + 9y = 0$, $y(0) = 1$, $y'(0) = b$.

Solution: Solving the ODE gives us $4r^2 + 12r + 9 = 0 \Rightarrow r = -3/2 \Rightarrow y = (c_1 + c_2 x)e^{-3x/2}$. The first initial condition gives $y(0) = c_1 = 1$. The other one gives $y'(0) = -3/2 + c_2 = b \Rightarrow c_2 = b + 3/2$. So, when $b < -3/2$ it's eventually negative, but when $b \geq -3/2$ it's always positive.

Ex: $y'' - 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 2$.

Solution: Again, we have to solve an IVP. Our roots are $r^2 - 6r + 9 = 0 \Rightarrow r = 3$. So, our solution is $y = (c_1 + c_2 x)e^{3x}$. From the initial conditions we have, $y(0) = c_1 = 0$ and $y'(0) = c_2 = 2$, so our solution is $y = 2xe^{3x}$.