3.2 Reduction of Order

The method we used in the beginning of class is called *reduction of order*, but we'll see that this is far more powerful than it first seemed. Consider the ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

and suppose we know one solution, say $y = y_1(x)$, then we "guess" the full solution is of the form $y = v(x)y_1(x)$. First we find the derivatives

$$y' = y_1'v + v'y_1 \Rightarrow y'' = y_1''v + 2v'y_1' + v''y_1.$$

Plugging this in and grouping the respective v's gives us

$$\begin{aligned} y_1''v + 2v'y_1' + v''y_1 + py_1'v + pv'y_1 + qy_1v &= y_1v'' + (2y_1' + py_1)v' + \cancelto{0}{(y_1'' + py_1' + qy_1)}v \\ &= y_1v'' + (2y_1' + py_1)v' = 0. \end{aligned}$$

And set $u = v'$ to get

$$\begin{aligned} &y_1u' + (2y_1' + py_1)u = 0 \Rightarrow u' = -\frac{2y_1' + py_1}{y_1}u = 0 \Rightarrow \int \frac{du}{u} = -\int \frac{2y_1' + py_1}{y_1}dx \\ &\Rightarrow \ln u = -\int \frac{2y_1' + py_1}{y_1}dx \Rightarrow u = \exp\left(-\int \frac{2y_1' + py_1}{y_1}dx\right) \\ &\Rightarrow v = \int \exp\left(-\int \frac{2y_1' + py_1}{y_1}dx\right). \end{aligned}$$

Now, we'll do some problems

Ex: $xy'' - y' + 4x^3y = 0$, $x > 0$; $y_1(x) = \sin x^2$.

Solution: Let $y = vy_1 \Rightarrow x(v''y_1 + 2v'y_1' + vy_1'') - (v'y_1 + vy_1') + 4x^3y_1 = 0$. Grouping all the v, v', and v'' terms gives

$$xy_1v'' + 2xy_1'v' - y_1v' + \cancelto{0}{(xy_1'' - y_1' + 4x^3y_1)}v = xy_1v'' + 2xy_1'v' - y_1v' = 0.$$

Set $u = v'$, then

$$\begin{aligned} &u' + \left(\frac{2y_1'}{y_1} - \frac{1}{x}\right)u = 0 \Rightarrow u' = \left(\frac{1}{x} - \frac{4x\cos x^2}{\sin x^2}\right)u \\ &\Rightarrow \ln u = \ln x - 2\int \frac{2x\cos x^2}{\sin x^2}dx = \ln x - \ln \sin^2 x^2 + C_0 \Rightarrow u = \frac{kx}{\sin^2 x^2} \\ &\Rightarrow v = k\int x\csc^2 x^2 dx = k_1 \cot x^2 + C_1 \Rightarrow y = k_1\cos x^2 + C_1 \sin x^2. \end{aligned}$$

So, $y_2 = \cos x^2$.

Ex: $x^2y'' - (x - 0.1875)y = 0$, $x > 0$; $y_1(x) = x^{1/4}e^{2\sqrt{x}}$.

Solution: Again we let $y = vy_1 \Rightarrow x^2(v''y_1 + 2v'y_1' + vy_1'') - (x - 0.1875)vy_1 = 0$. Grouping gives

$$x^2y_1v'' + 2x^2y_1'v' + \cancelto{0}{[x^2y_1'' - (x - 0.1875)y_1]}v = x^2y_1v'' + 2x^2y_1'v' = 0.$$

Set $u = v'$, then

$$\begin{aligned} &u' = -2\frac{y_1'}{y_1}u = \left(\frac{-2}{\sqrt{x}} - \frac{1}{2x}\right)u \Rightarrow \ln u = -2\int x^{-1/2}dx + \frac{1}{2}\int \frac{dx}{x} = -4\sqrt{x} - \frac{1}{2}\ln x + C \\ &\Rightarrow u = \frac{k}{\sqrt{x}}e^{-4\sqrt{x}} \Rightarrow v = ke^{-4\sqrt{x}} + C \Rightarrow y = kx^{1/4}e^{-2\sqrt{x}} + Cx^{1/4}e^{2\sqrt{x}}. \end{aligned}$$

So, $y_2 = x^{1/4}e^{-2\sqrt{x}}$