3.4 Undetermined Coefficients

Consider the nonhomogeneous ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_2 y'' + a_1 y' + a_0 = f(x). \tag{1}$$

Notice that our usual solution won't work, but maybe it's part fo the solution. Suppose y_p is a solution to (1) that is linearly independent with respect to the solution the homogeneous ODE. Let y be the general solution of (1). Lets plug in $y_c = y - y_p$ in (1), then we get that y_c is a solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots a_2 y'' + a_1 y' + a_0 = 0. \tag{2}$$

So, in fact y_c is the solution to the homogeneous ODE, so $y = y_c + y_p$, where y_c is the homogeneous part of the solution and y_p is the purely nonhomogeneous part of the solution.

Definition 1. The characteristic solution, y_c, is the general solution of (2) and the particular solution, y_p, is the additional solution to (1).

Case1: No term in $f(x)$ is the same as any term in y_c. Then y_p is a linear combination of terms of $f(x)$ and their derivatives.

- $f_1(x) = x^n \Rightarrow y_{1_p} = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$. If our f is a polynomial, the particular solution will be of the form of the most general polynomial of order of the polynomial in f.
- $f_2(x) = e^{mx} \Rightarrow y_{2_p} = k e^{mx}$. This one is easy.
- $f_3(x) = \cos(mx)$ or $\sin(mx)$, then $y_{3_p} = A\cos(mx) + B\sin(mx)$. If we have sine or cosine our particular solution will be a linear combination of sines and cosines.
- $f(x) = f_1(x) + f_2(x) + f_3(x) \Rightarrow y_p = y_{1_p} + y_{2_p} + y_{3_p}$. If we have a combination of these simple examples then we just combine all of their respective particular solutions.
- $f(x) = f_1(x) f_2(x) f_3(x) \Rightarrow y_p = y_{1_p} y_{2_p} y_{3_p}$. We do the same sort of thing with products.

Case 2: $f(x)$ contains terms that are x^n times terms in y_c, i.e. if $u(x)$ is a term of y_c and $f(x)$ contains $x^n u(x)$. Then y_p is as usual but multiply by "x".

- Consider $y_c = g(x) + e^{mx}$ and $f(x) = l(x) + x^n e^{mx}$, where we don't care about $g(x)$ and $l(x)$ – we are just thinking of them as place holders. Then our particular solution is $y_p = h(x) + (A_n x^{n+1} + A_{n-1} x^n + \cdots + A_0 x) e^{mx}$.
- Consider a similar case except with sine, also equivalently cosine. $y_c = g(x) + \sin(mx)$ and $f(x) = l(x) x^n \sin(mx)$, then our particular solution is, $y_p = h(x) + (A_n x^{n+1} + A_{n-1} x^n + \cdots + A_0 x)(B\cos(mx) + C\sin(mx))$.

Case 3: If y_c contains repeated roots with the highest being of order λ (i.e. x^λ) and $f(x)$ contains terms x^n times the repeated root terms, then multiply out by $x^{\lambda+1}$.

- $y_c = g(x) + x^\lambda + \cdots + e^{mx}$ and $f(x) = l(x) + x^n e^{mx}$, then our particular solution is $y_p = h(x) + x^{\lambda+1}(A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0) e^{mx}$.

The idea for the repeated cases is to get rid of all the repeats while preserving the same amount of constants. The cases that are outlined above are very general, so I made a table of the type of expressions we would most likely come across

Case	Characteristic Solution	Repeat	Form of Particular Solution
Case 2	$y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	$x^n e^{r_1 x}$	$y_p = x(A_n x^n + \cdots + A_x + A_0) e^{r_1 x}$
	$y_c =^{\xi x} (A\cos(\theta x) + B\sin(\theta x))$	$x^n e^{\xi x}\cos(\theta x)$	$y_p = x(A_n x^n + \cdots + A_x + A_0) e^{\xi x}\cos(\theta x)$
Case 3	$y_c = (c_1 + c_2 x) e^{\lambda x}$	$x^n e^{\lambda x}$	$y_p = x^2(A_n x^n + \cdots + A_x + A_0) e^{\lambda x}$

It can be tricky to figure out what y_p has to be in the beginning, but hopefully some practice problems will help us.

Ex: $y'' + 2y' = 3 + 4\sin 2t$.

Solution: $r^2 + 2r = r(r+2) = 0 \Rightarrow r = 0, -2$, so $y_c = c_1 + c_2 e^{-2t}$. Since $f(t) = 3 + 4\sin 2t$, our initial guess for the particular solution is $y_p = A + B\cos 2t + C\sin 2t$, but this would be incorrect because we already have a lone constant in our characteristic solution, so our actual particular solution is $y_p = At + B\cos 2t + C\sin 2t$. Plugging this into the ODE gives,

$$4(C-B)\cos 2t - 4(B+C)\sin 2t + 2A = 3 + 4\sin 2t.$$

Matching the terms gives $2A = 3 \Rightarrow A = 3/2$ immediately. From the cosine term we get $4(C-B) = 0 \Rightarrow C = B$ because there is no cosine term on the right hand side. From the sine terms we have $-4(B+C) = 8B = 4 \Rightarrow C = B = -1/2$, so our particular solution is $y_p = \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$. Then our general solution is

$$y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t.$$

Ex: $y'' + 9y = t^2 e^{3t} + 6$.

Solution: $r^2 + 9 = 0 \Rightarrow r = \pm 3i$, then $y_c = A\cos 3t + B\sin 3t$. Since $f(t) = t^2 e^{3t} + 6$, $y_p = (At^2 + Bt + C)e^{3t}$, and there are no repeats. Plugging this into the ODE gives

$$2Ae^{3t} + 6(2At + B)e^{3t} + 18(At^2 + Bt + C)e^{3t} + 9D = t^2 e^{3t} + 6$$
$$\Rightarrow 18At^2 e^{3t} + (12A + 18B)te^{3t} + (2A + 6B + 18C)e^{3t} + 9D = t^2 e^{3t} + 6.$$

Matching terms immediately gives us $9D = 6 \Rightarrow D = 2/3$. From the $t^2 e^{3t}$ we get $18A = 1 \Rightarrow A = 1/18$. The other terms are zero so we get, $12/18 + 18B = 0 \Rightarrow B = -1/27$, and $1/9 + 2/9 + 18C = 0 \Rightarrow C = 1/162$. So, our particular solution is $y_p = (t^2/18 - t/27 + 1/162)e^{3t} + 2/3$. Then our general solution is

$$y = A\cos 3t + B\sin 3t + \left(\frac{1}{18}t^2 - \frac{1}{27}t^2 + \frac{1}{162}\right)e^{3t} + \frac{2}{3}.$$

Ex: $y'' - 2y' - 3y = 3te^{2t}$; $y(0) = 1$, $y'(0) = 0$.

Solution: $r^2 - 2r - 3 = (r-3)(r+1) = 0 \Rightarrow r = 3, -1 \Rightarrow y_c = c_1 e^{3t} + c_2 e^{-t}$. Since $f(t) = 3te^{2t}$, $y_p = (At + B)e^{2t}$ and there are no repeats. Plugging this into the ODE gives

$$4Ae^{2t} + 4(At+B)e^{2t} - 2Ae^{2t} - 4(At+B)e^{2t} - 3(At+B)e^{2t} = -3Ate^{2t} + (2A - 3B)e^{2t} = 3te^{2t}.$$

Matching the te^{2t} gives $-3A = 3 \Rightarrow A = -1$. The other term is zero, so we get $-2 - 3B = 0 \Rightarrow B = -2/3$. This gives us $y_p = (-t - 2/3)e^{2t}$, then our general solution is

$$y = c_1 e^{3t} + c_2 e^{-t} + \left(-t - \frac{2}{3}\right)e^{2t}.$$

The first initial condition gives $y(0) = c_1 + c_2 - 2/3 = 1 \Rightarrow c_1 + c_2 = 5/3$, and he second gives $y'(0) = 3c_1 - c_2 - 1 - 4/3 = 0 \Rightarrow 3c_1 - c_2 = 7/3$. Now we add the equations to get $4c_1 = 4 \Rightarrow c_1 = 1 \Rightarrow c_2 = 2/3$. Then our solution is

$$y = e^{3t} + \frac{2}{3}e^{-t} + \left(-t - \frac{2}{3}\right)e^{2t}.$$

Ex: $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$.

Solution: For this problem we only need the form of the particular solution. In order to get that we still have to compute the characteristic solution: $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$, which gives $y_c = e^{-t}(c_1\sin t + c_2\cos t)$. From $f(x)$ we can guess a particular solution of

$$y_p \stackrel{?}{=} e^{-t}[A + B\cos t + C\sin t + (D_2t^2 + D_1t + D_0)\cos t + (E_2t^2 + E_1t + E_0)\sin t]$$
$$\stackrel{?}{=} e^{-t}[A + (B_2t^2 + B_1t + B_0)\cos t + (C_2t^2 + C_1t + C_0)\sin t]$$

However, this would be wrong due to the repeats. So, we need to multiply the cosine and sine block out by t

$$y_p = e^{-t}[A + t(B_2t^2 + B_1t + B_0)\cos t + t(C_2t^2 + C_1t + C_0)\sin t]$$