For this chapter lets go through all the material first, and then do problems.

Definition 1. If $A \neq \emptyset$ and $B \neq \emptyset$, then the Cartesian product, $A \times B$, contains all (a, b) such that $a \in A$ and $b \in B$; i.e. $A \times B := \{(a, b) : a \in A, b \in B\}$.

For example $A = \{1, 2, 3\}, B = \{4, 5\} \Rightarrow A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

Definition 2. A function $f : A \mapsto B$ is a set of ordered pairs $(a, b) \in A \times B$ such that for all $a \in A$, there is a unique $b \in B$; i.e. if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. The set A is said to be the domain and $f(A)$ is said to be the range. The set B is said to be the codomain and $f(A) \subseteq B$.

We discussed examples of this in class.

Definition 3. Consider $f : A \mapsto B$. If $E \subseteq A$, then the image of E under f is $f(E) := \{f(x) : x \in E\} \subseteq B$. Further, if $H \subseteq B$, then the inverse image of H under f is $f^{-1}(H) := \{x \in A : f(x) \in H\} \subseteq A$, where f^{-1} is called the inverse.

Definition 4. Consider $f : A \mapsto B$.
  (1) If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, then f is said to be injective (one-to-one) and we call f an injection.
  (2) If $f(A) = f(B)$, then f is said to be surjective (onto) and we call f a surjection.
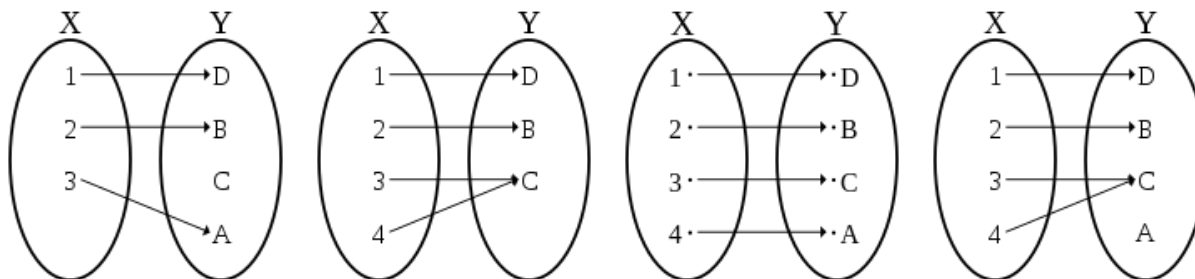 (3) If f is both injective and surjective, it is said to be bijective and called a bijection.

FIGURE 1. Injection, Surjection, Bijection, Neither

Example 1. *The function $f : (0, \infty) \mapsto (0, 1)$ defined as*

$$f := \frac{1}{1 + x^2} \tag{1}$$

is a bijection.

The strategy for injection is to show $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, and the strategy for surjection is to show for all $y \in (0, 1)$ there is an $x \in (0, \infty)$ such that $f(x) = y$.

Proof. We do the proof in two parts,

Injection: Suppose $f(x_1) = f(x_2)$, then

$$\frac{1}{1 + x_1^2} = \frac{1}{1 + x_2^2} \Rightarrow 1 + x_1^2 = 1 + x_2^2 \Rightarrow x_1^2 = x_2^2,$$

and since $x_1, x_2 \in (0, \infty)$; i.e. positive, $x_1 = x_2$.

Surjection: Consider $y \in (0, 1)$ then if $f(x) = y$, $x = \sqrt{(1/y) - 1} \in (0, \infty)$.

Since the function is both an injection and a surjection, it is a bijection.

$\square$

Definition 5. If $f : A \mapsto B$ is bijective, then the inverse is $f^{-1} := \{(b, a) \in B \times A : (a, b) \in f\}$.

For example, for $f : (0, \infty) \mapsto (0, 1)$, where $f(x) := 1/(1 + x^2)$, we write $x = 1/(1 + y^2)$ and solve for y, which gives us $y = \sqrt{1/x - 1}$, which means $f^{-1}(x) = \sqrt{1/x - 1}$.

Definition 6. Consider $f : A \mapsto B$ and $g : B \mapsto C$, then the composition $g \circ f : A \mapsto C$ is defined as $(g \circ f)(x) := g(f(x))$ for all $x \in A$.

Now lets do a bunch of exercises from the book before tackling a theorem.

10.4 (a) This is clearly a function. Suppose it wasn't, then we would have two values of y, say y_1 and y_2 for a single value of x. However, if $y_1 = 4x - 3$ and $y_2 = 4x - 3$, then they are equal since $4x - 3$ is unique.

   (b) Counterexample: $x = 1$, $y = -1, -3$.

10.6 (a) $A = \mathbb{R}$, $f(A) = [1, \infty)$.

   (b) $A = \mathbb{R} \setminus \{0\}$, $f(A) = \mathbb{R} \setminus \{1\}$.

10.9c) If $3a + 5b = 1$, $b = (1 - 3a)/5$ yields a unique rational solution.

10.11 (a) $f(C) = C$, $f^{-1}(C) = \mathbb{R} \setminus (-1, 1)$, $f^{-1}(D) = \mathbb{R}$, and $f^{-1}(\{1\}) = \pm 1$.

   (b) $f(C) = \mathbb{R}^+$, $f^{-1}(C) = [e, \infty)$, $f^{-1}(D) = [1, \infty)$, and $f^{-1}(\{1\}) = e$.

10.12 (a) *Proof.* Suppose $x \in C \cup D$ and $y = f(x) \in f(C \cup D)$, then $x \in C$ or $x \in D$. If $x \in C$, then $y \in f(C)$ or if $x \in D$, $y \in f(D)$. Therefore, $y \in f(C)$ or $y \in f(D)$; i.e., $y \in f(C) \cup f(D)$, then $f(C \cup D) \subseteq f(C) \cup f(D)$.

   Now suppose $y \in f(C) \cup f(D)$, then $y \in f(C)$ or $y \in f(D)$. So if $y = f(x)$, $x \in C$ or $x \in D$; i.e., $x \in C \cup D$. Hence $y = f(x) \in f(C \cup D)$ and $f(C) \cup f(D) \subseteq f(C \cup D)$.

   Therefore, $f(C \cup D) = f(C) \cup f(D)$. $\qquad\square$

   (e) *Proof.* Suppose $x \in E \cap F$ and $y \in f^{-1}(E \cap F)$. Then, $x \in E$ and $x \in F$, so $y \in f^{-1}(E)$ and $y \in f^{-1}(F)$; i.e., $y \in f^{-1}(E) \cap f^{-1}(F)$. Therefore, $f^{-1}(E \cap F) \subseteq f^{-1}(E) \cap f^{-1}(F)$.

   Now suppose $y \in f^{-1}(E) \cap f^{-1}(F)$, then $y \in f^{-1}(E)$ and $y \in f^{-1}(F)$. So, $x \in E$ and $x \in F$; i.e., $x \in E \cap F$. Hence, $y = f(x) \in f(E \cap F)$, and $f^{-1}(E) \cap f^{-1}(F) \subseteq f^{-1}(E \cap F)$.

   Therefore, $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$ $\qquad\square$

10.20 (a) Suppose $f(n_1) = f(n_2) \Rightarrow 2n_1 + 1 = 2n_2 + 1 \Rightarrow n_1 = n_2$. ✓

   (b) Suppose $m \in \mathbb{Z}$, and $m = 2n + 1 \Rightarrow n = (m - 1)/2$, which may not be an integer so this function is not surjective.

10.30 Here we must prove that it is both injective and surjective.

   Proof. Injective: Suppose $f(x_1) = f(x_2) \Rightarrow \cancel{7}x_1 \cancel{-2} = \cancel{7}x_2 \cancel{-2}$ ✓
   Surjective: Suppose $y \in \mathbb{R}$ and $y = 7x - 2 \Rightarrow x = (y + 2)/7 \in \mathbb{R}$ ✓
   Therefore, since the function is both injective and surjective, it is bijective.

   $\qquad\square$

10.38 Notice that

$$g(f(x)) = 5(3x^2 + 1) - 3 \Rightarrow g(f(1)) = 17.$$
$$f(g(x)) = 3(5x - 3)^2 + 1 \Rightarrow f(g(1)) = 13.$$

10.42a) *Proof.* Surjective: since f is surjective, $f(A) = B$, and since g is surjective $g(B) = C$, then for $g \circ f : A \mapsto C$, $g(f(A)) = g(B) = C$. ✓
   Injective: Since g is injective, if $g(y_1) = g(y_2)$, $y_1 = y_2$, and since f is injective $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then if $g \circ f(x_1) = g \circ f(x_2)$, $g(f(x_1)) = y_1 = g(f(x_2)) = g(y_2)$. Since $y_1 = y_2$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. $\qquad\square$

10.50) We need to first show that this is bijective, then we can take the inverse.

Proof. Injective: Suppose $f(x_1) = f(x_2) \Rightarrow \cancel{4}x_1 \cancel{-3} = \cancel{4}x_2 \cancel{-3}$. ✓
Surjective: Consider $y \in \mathbb{R}$, then if $y = 4x - 3$, $x = (y+3)/4 \in \mathbb{R}$.
Therefore, it is a bijective function. □

Now we are allowed to take the inverse, which is $f^{-1}(x) = (x+3)/4$.

10.32) Here lets show its bijective, which the problem asks us to do, but then we will compute the inverse.

Proof. Injective: Suppose $f(x_1) = f(x_2)$, then

$$\frac{5x_1 + 1}{x_1 - 2} = \frac{5x_2 + 1}{x_2 - 2} \Rightarrow (5x_1 + 1)(x_2 - 2) = (5x_2 + 1)(x_1 - 2)$$

because $x_1, x_2 \in \mathbb{R} \setminus \{2\}$. Therefore,

$$\cancel{5x_1 x_2} - 10x_1 + x_2 \cancel{-2} = \cancel{5x_1 x_2} - 10x_2 + x_1 \cancel{-2} \Rightarrow x_1 = x_2. \qquad ✓$$

Surjective: Consider $y \in \mathbb{R} \setminus \{5\}$, then if $y = (5x + 1)/(x - 2)$,

$$yx - 2y = 5x + 1 \Rightarrow x(y - 5) = 1 + 2y \Rightarrow x = \frac{1 + 2y}{y - 5} \in \mathbb{R} \setminus \{2\},$$

because

$$\lim_{y \to \pm\infty} \frac{1 + 2y}{y - 5} = \frac{2}{1} = 2. \qquad ✓$$

Therefore, it is a bijective function. □

Now we are allowed to take the inverse, which is $f^{-1}(x) = (1 + 2x)/(x - 5)$.

Theorem 1. *Consider $f : A \mapsto B$ and $g : B \mapsto C$, and $H \subseteq C$, then $(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$.*

Here, instead of trying to prove one is a subset of the other and vice-versa, we shall use a more direct approach using the definition of the inverse, which gives us $h(h^{-1}(x)) = x$.

Proof. Notice that by definition of the inverse (first) and the composition (second), $(g \circ f)((g \circ f)^{-1}(H)) = g(f((g \circ f)^{-1}(H))) = H$. Therefore, $f^{-1}(g^{-1}(H)) = f^{-1}(g^{-1}(g(f((g \circ f)^{-1}(H))))) = f^{-1}(f((g \circ f)^{-1}(H))) = (g \circ f)^{-1}(H)$. □

Caveat: You may be tempted to work on both sides of the equation $(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$ simultaneously to massage it into a statement that is true. However, this would mean that you are already assuming the statement you are trying to prove is true (this type of fallacy is called "Begging the question"). What you must do is go from a true statement and massage that into the statement you would like to prove.