Definition 1. We call the amount of elements in a set A the cardinality of A, denoted as Card(A).

Properties of infinite sets.

- If C is an infinite set and B is a finite set, then $C \setminus B$ is infinite.
- If $A \subseteq B$ is an infinite set, then so is B.
- $\mathbb{N}$ is said to be denumerable (countably infinite). Card($\mathbb{N} = \aleph_0$)

Definition 2. A set S is said to be countably infinite if there is a bijection $f : \mathbb{N} \mapsto S$. A set is countable if it is finite or countably infinite. A set that is not countable is said to be uncountable.

It should be noted that I generally call countably infinite sets simply countable since finite sets are trivially countable.

Here are some examples below:

- $S = \{1, 4, 9, 16, \ldots\}$ is countable since $f : \mathbb{N} \mapsto S$ with the mapping $f(n) = n^2$ for all $n \in \mathbb{N}$.
- $\mathbb{Z}$ is countable since $f : \mathbb{N} \mapsto \mathbb{Z}$ such that $f(n) = n/2$ for all even $n \in \mathbb{N}$, $f(n) = -(n-1)/2$ for all odd $n \in \mathbb{N}$, and $f(1) = 0$.
- Any sequence $A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is countable because by definition a sequence is a function for $\mathbb{N}$ onto the elements of the sequence; i.e. $f : \mathbb{N} \mapsto A$ such that $f(n) = a_n$.
- A finite union of countable sets are countable. Consider $B = \{b_1, b_2, b_3, \ldots, b_n, \ldots\}$. Then $f : \mathbb{N} \mapsto A \cup B$ such that $f(n) = a_{n/2}$ for $n \in \mathbb{N}$ even and $f(n) = b_{(n+1)/2}$ for $n \in \mathbb{N}$ odd.
- We can also take Cartesian products of countable sets to make countable sets. $f : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ such that $f^{-1}(n, m) := \frac{1}{2}(m + n - 2)(m + n - 1) + m$.
- For the irrationals it gets a bit tricky, but notice that $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{Q}^+$ such that $f(n, m) = n/m$ for all $n, m \in \mathbb{N}$ by definition. We can do the same thing for $\mathbb{Q}^-$ by $f(n, m) = -n/m$. And $\{0\}$ is finite, so that is already countable. Therefore $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$.

Here are some other properties of infinite sets

- If $T \subseteq S$, If S is countable then so is T and if T is uncountable then so is S.
- If A_m is countable $A := \cup_{m=1}^{\infty} A_m$ is countable.
- Also, bijections are not always necessary. Both $f : \mathbb{N} \mapsto S$ (surjective) and $f : S \mapsto \mathbb{N}$ (injective) also work. This is because an infinite subset of $\mathbb{N}$ had the same number of elements as $\mathbb{N}$ as we showed in our first example. So a subset of $\mathbb{N}$ works just as well as $\mathbb{N}$.

For the exam you just have to know that if a function is a bijection between two sets, those sets have the same cardinality.