14.1 Limits of sequences

Definition 1. A sequence $\{x_n\} \subseteq \mathbb{R}$ converges if there is an $x \in \mathbb{R}$ such that
For every $\varepsilon > 0$, there is an $\mathbb{N}$ such that $|x - x_n| \leq \varepsilon$ for all $n \geq \mathbb{N}$;
otherwise it diverges. We call this x the limit of $\{x_n\}$.

Notation: $\lim_{n\to\infty} x_n = x$, $x_n \to x$ as $n \to \infty$, or just $x_n \to x$.

Theorem 1 (Triangle Inequality)**.** *For $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.*

Proof. Since $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$, $-(|x| + |y|) \leq x + y \leq |x| = |y|$, then $|x + y| \leq |x| + |y|$. □

Theorem 2 (Uniqueness)**.** *A sequence $\{x_n\} \subseteq \mathbb{R}$ has at most one limit.*

Proof. Assume $x_n \to p$ and $x_n \to q$. By the triangle inequality, $|p-q| \leq |p-x_n|+|q-x_n|$. Since $|p-x_n| \to 0$ and $|q - x_n| \to 0$, $|p - q| \to 0 \Rightarrow p = q$. □

Now lets do a few exercise problems from the book.

14.3) **Scratch work:** First lets do some scratch work. We want to show

$$\left|\frac{1}{2n} - 0\right| \leq \varepsilon,$$

then if $1/2n \leq \varepsilon$, $n \geq 1/2\varepsilon$. Now we can do the proof.

Proof. For every $\varepsilon > 0$, choose $N(\varepsilon) = 1/2\varepsilon$, then for all $n \geq N$,

$$\left|\frac{1}{2n} - 0\right| \leq \varepsilon \tag{1}$$

□

14.5) **Scratch work:** Again we will do some scratch work. We want to show

$$\left|\left(1 + \frac{1}{2^n}\right) - 1\right| = \left|\frac{1}{2^n}\right| \leq \varepsilon,$$

then since $n > 0$, $1/2^n \leq \varepsilon$. Cross multiplying and taking the log of both sides gives us

$$2^n \geq \frac{1}{\varepsilon} \Rightarrow n \ln 2 \geq \ln\left(\frac{1}{\varepsilon}\right) \Rightarrow n \geq \frac{\ln(1/\varepsilon)}{\ln 2}.$$

Now we can write the proof

Proof. For every $\varepsilon > 0$, choose $N(\varepsilon) = \frac{\ln(1/\varepsilon)}{\ln 2}$, then for all $n \geq N$,

$$\left|\left(1 + \frac{1}{2^n}\right) - 1\right| = \left|\frac{1}{2^n}\right| \leq \varepsilon \tag{2}$$

□

Now lets do a few example problems for finding the limit of sequences

Ex: Prove that

$$\frac{1}{n} - \frac{1}{n+1} \to 0 \tag{3}$$

Solution: First lets do some back of the envelope calculations. We want to show

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| < \varepsilon,$$

so lets first see if we can bound our sequence by something easier to deal with

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n(n+1)}\right| < \left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n}\right|.$$

Now we can do the proof

Proof. For every $\varepsilon > 0$, choose $N(\varepsilon) = 1/\varepsilon$, then for all $n \geq 1/\varepsilon$,

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| < \frac{1}{n} \leq \varepsilon \tag{4}$$

□

Ex: Prove that

$$(2n)^{1/n} \to 1. \tag{5}$$

Solution: Again we do some back of the envelope calculations. Notice that

$$(2n)^{1/n} = e^{\ln(2n)/n}$$

so

$$\left|\frac{\ln(2n)}{n}\right| < \varepsilon_1 \Rightarrow \left|e^{\ln(2n)/n} - 1\right| < \varepsilon$$

if $\varepsilon_1 = \ln(\varepsilon + 1)$. Furthermore,

$$\left|\frac{\ln(2n)}{n}\right| < \frac{1}{\ln(n)} \qquad \text{for } n > 1.$$

and $1/\ln(n) < \varepsilon_1$ for $n > e^{1/\varepsilon_1}$. So, lets choose

$$N(\varepsilon) = e^{1/\ln(\varepsilon+1)}.$$

Proof. For every $\varepsilon > 0$, choose $N(\varepsilon) = e^{1/\ln(\varepsilon+1)}$, then for all $n \geq e^{1/\ln(\varepsilon+1)}$,

$$\left|\frac{\ln(2n)}{n}\right| < \frac{1}{\ln(n)} \leq \ln(\varepsilon + 1) \Rightarrow \left|(2n)^{1/n} - 1\right| = \left|e^{\ln(2n)/n} - 1\right| < \left|e^{\ln(\varepsilon+1)} - 1\right| = \varepsilon.$$

□

Ex: Prove that $r^n \to 0$ if $|r| < 1$.

Solution: Notice that $|r|^n = \exp(n \ln |r|)$, so if $n \ln |r| = \ln(\varepsilon)$, then $n = \ln(\varepsilon)/\ln |r|$. Clearly this n doesn't work, but it gives us a starting point.

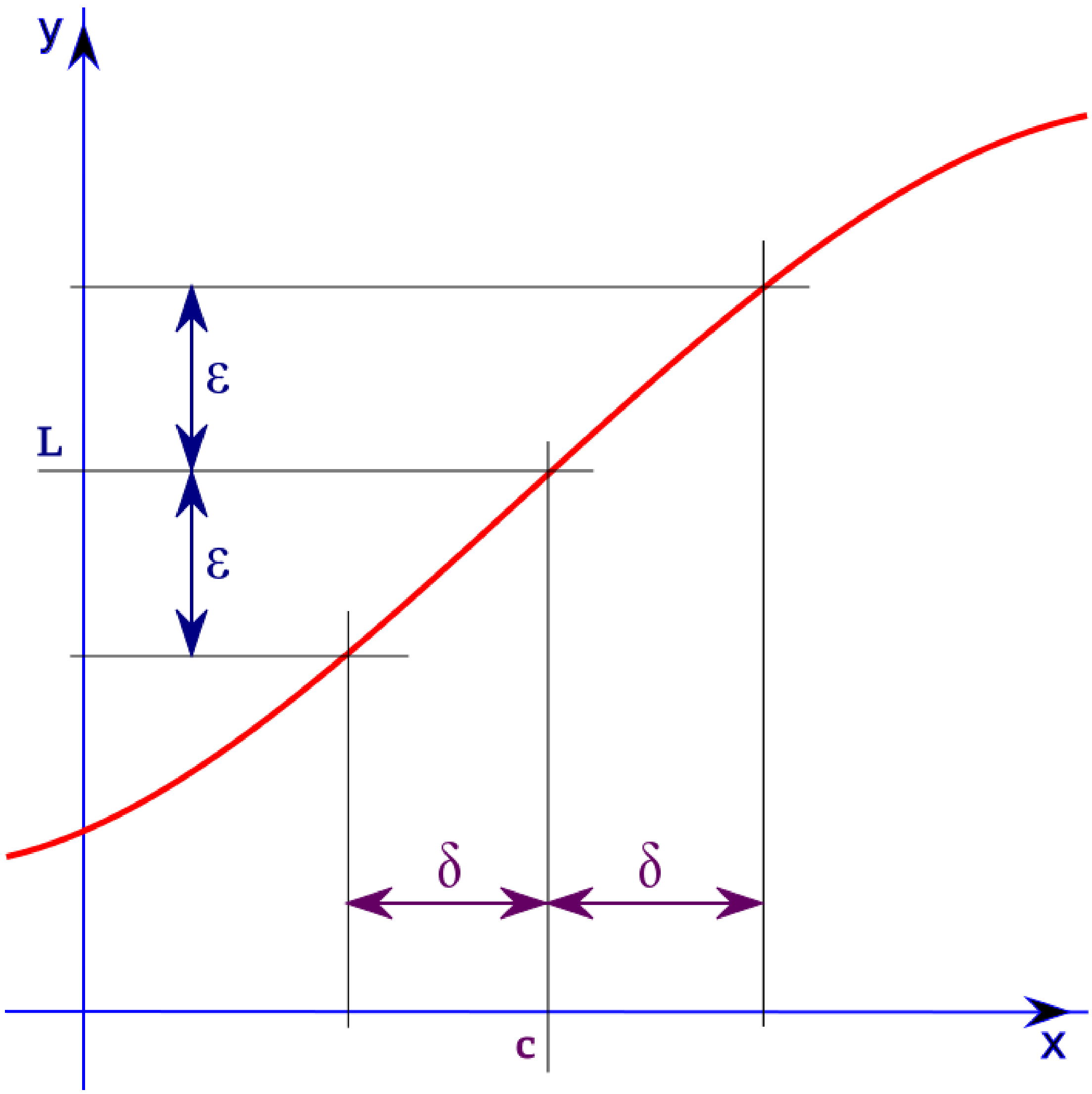
Proof. For every $\varepsilon > 0$, choose $N(\varepsilon) > \ln \varepsilon - \ln |r|$, then for all $n \geq N$,

$$|r^n| = \big|\exp(n \ln |r|)\big| < \left|\exp\left(\frac{\ln \varepsilon}{\ln |r|} \ln |r|\right)\right| = \varepsilon$$

□

Definition 2. A function $f : A \to \mathbb{R}$ has a limit L near $a \in A$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$, $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

This is illustrated in the figure below.



Now lets look at some exercise problems from the book.

14.18) **Scratch work:** We have that $|x - 2| < \delta$ and we want $|(3x/2 + 1) - 4| < \varepsilon$. We write

$$\left|\frac{3}{2}x + 1 - 4\right| = \left|\frac{3}{2}x - 3\right| = \frac{3}{2}|x - 2| < \frac{3}{2}\delta.$$

Now we can write the proof.

Proof. Choose $\delta = 2\varepsilon/3$, then for all $\varepsilon > 0$, we have

$$|x-2| < \delta \Rightarrow \left|\frac{3}{2}x + 1 - 4\right| = \left|\frac{3}{2}x - 3\right| = \frac{3}{2}|x-2| < \frac{3}{2}\delta = \varepsilon. \tag{6}$$

□

14.20) **Scratch work:** We have that $|x-2| < \delta$ and we want $|(2x^2 - x - 5) - 1| < \varepsilon$. We write

$$|2x^2 - x - 6| = |2x+3||x-2| < |2x+3|\delta$$

Notice that our coefficient here is not constant, so we must bound it. If $\delta = 1$ we would have that $|2x+3| < 5$, then $|2x^2 - x - 6| < |2x+3|\delta < 5\delta = \varepsilon$. Now we can write our proof

Proof. Choose $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$, then for all $\varepsilon > 0$, we have

$$|x-2| < \delta \Rightarrow |2x^2 - x - 6| = |2x+3||x-2| < \varepsilon. \tag{7}$$

□

14.22) **Scratch work:** We have that $|x-1| < \delta$ and we want $|1/(5x-4) - 1| < \varepsilon$. We write

$$\left|\frac{1}{5x-4} - 1\right| = \left|\frac{1-(5x-4)}{5x-4}\right| = \left|\frac{5-5x}{5x-4}\right| = \left|\frac{5}{5x-4}\right||x-1| < \left|\frac{5}{5x-4}\right|\delta$$

This is a bit tricky because we need to avoid the point $x = 4/5$, but it gets even worse because not only do we have to avoid it, we also have to be bounded away from it. So we cannot pick $\delta = 1/5$, we must go with something smaller, say $\delta = 1/10$.

If $\delta = 1/10$, then

$$\left|\frac{5}{5x-4}\right| \leq 10 \Rightarrow \left|\frac{5}{5x-4}\right|\delta \leq 10\delta = \varepsilon.$$

Now we may write our proof.

Proof. Choose $\delta = \min(1/10, \varepsilon/10)$, then for all $\varepsilon > 0$, we have

$$|x-1| < \delta \Rightarrow \left|\frac{1}{5x-4} - 1\right| = \left|\frac{5}{5x-4}\right||x-1| < \varepsilon$$

□

Now lets look at a few easy examples.

Ex: If $f(x) = x$, prove that $\lim_{x\to 1} f(x) = 1$.

Proof. Choose $\delta = \varepsilon$, then for all $\varepsilon > 0$ we have

$$|x-1| < \delta \Rightarrow |f(x) - 1| = |x-1| < \delta = \varepsilon.$$

□

Ex: If $f(x) = x^2$, prove that $\lim_{x\to 2} f(x) = 4$.

Scratch Work: We have that $|x-2| < \delta$ and we want $|x^2 - 4| < \varepsilon$. Notice that if $|x-2| < 1$, then we can bound $|x+2| < 5$ because the supremum of x can be in that neighborhood is 3. Then we have that $|x^2 - 4| = |x-2||x+2| < 5|x-2|$. However we need the added requirement $|x-2| < \varepsilon/5$.

Proof. Choose $\delta = \min(1, \varepsilon/5)$, then for all $\varepsilon > 0$ we have

$$|x-2| < \delta \Rightarrow |f(x) - 4| = |x^2 - 4| = |x-2||x+2| < 5|x-2| < \varepsilon.$$

□