5.1 Counterexamples

Almost always, a result worth mentioning will hold for infinitely many values for x. However, if this result is false, we need only disprove it at a single point. This is called a counterexample.

Again its best to actually look at examples.

5.1) log does not exist for negative values.

5.3) If $n = 3$, $2n^2 + 1 = 19$, and 3 does not divide 19.

5.5) Notice that

$$(a+b)^2(a+b) = (a+b)(a^2+2ab+b^2) = a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 = a^3+3a^2b+3ab^2+b^3.$$

This only works if $2ab = a^2b + ab^2$, so we choose $a = 1$ and $b = 2$.

5.9) We may guess that this result may hold for some choice of x if $n = 2$. Then we may simply calculate what that value of x must be,

$$x^2+(x+1)^2 = (x+2)^2 \Rightarrow x^2+x^2+2x+1 = x^2+4x+4 \Rightarrow x^2-2x-3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3.$$

5.2 Proof by contradiction

It is not uncommon to try something and fail. But this may play to our advantage for proofs. Lets say we want to prove $P(x) \Rightarrow Q(x)$ where P is true. We disregard P is false because that is vacuous. Now, if we assume Q is false, but fail to prove it, then we will have proved that Q is true. This is called a proof by contradiction.

Lets look at the following example. A similar example is in the book for $\sqrt{2}$, which you should read and understand.

Theorem 1. *$\sqrt{3}$ is irrational.*

Proof. Suppose not; i.e., $\sqrt{3} \in \mathbb{Q}$. Then $\sqrt{3} = m/n$ such that $n, m \in \mathbb{N}$, and suppose this is in lowest form; i.e., n, m have no common factors. Then $m^2 = 3n^2$, so m is divisible by 3; i.e.,

$$m = 3k \Rightarrow 9k^2 = 3n^2 \Rightarrow n^2 = 3k^2,$$

so n is divisible by 3, but we assumed m and n have no common factors. This forms a contradiction. □

Now lets look at some simpler exercises from the book.

5.12) *Proof.* Suppose not, then there is an $x \in \mathbb{Q}$ such that $-m/n \leq x < 0$ for all $n, m \in \mathbb{N}$. However, $x/2 \in \mathbb{Q}$ and $x < x/2 < 0$. This forms a contradiction. □

5.14) *Proof.* Suppose not; i.e., $200 = 2k + 1 + 2m + 2n$ for $k, m, n \in \mathbb{Z}$. Then

$$200 = 2(k+m+n)+1 \Rightarrow 100 = k+m+n+1/2 \notin \mathbb{Z}.$$

This forms a contradiction. □

5.18) *Proof.* Suppose not. Let $x \in \mathbb{Q}^c$ and $m, r \in \mathbb{Q}$, then $r = mx$. Therefore, $x = r/m$, however the ratio of rational numbers is rational since by definition rational numbers are the ratio of integers, so x is also a ratio of integers. □

5.26) Suppose there is an $x \in \mathbb{Z}$ such that $2x < x^2 < 3x \Rightarrow 2 < x < 3$. This forms a contradiction.

Next we move on to Section 5.4, but if you are having trouble with proofs it may be useful to read through Section 5.3.

5.4 Existence proofs

This won't be on the first exam, but will be used with Calculus proofs, and since this is in this chapter, lets look at a few examples.

5.42) *Proof.* Choose $a = 0$. $\square$

The next couple of problems are quite interesting, but will never show up on any exam in this class due to their difficulty. However, if you want to continue in math you will need to get used to these types of ambiguous problems.

5.43) *Proof.* Notice that $2 \in \mathbb{Q}$ and $1/2\sqrt{2} \in \mathbb{Q}^c$. Then if $2^{1/2\sqrt{2}} \in Q^c$ we are done. If $2^{1/2\sqrt{2}} \in \mathbb{Q}$ then

$$\left(2^{1/2\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2} \in \mathbb{Q}^c.$$

Either way there is an a and b such that a^b is irrational. $\square$

5.44) *Proof.* If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ we are done. If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}^c$, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{2/\sqrt{2}} = 2$. Either way there is an a and b such that a^b is rational. $\square$

The next problem is something we will see when we do Calculus proofs.

5.46) *Proof.* First lets show existence. Notice that $p(2/3) = 8/27 + 4/9 - 1 = 20/27 - 1 < 1$ and $p(1) = 1 > 0$. Then by the Intermediate Value Theorem there is an x_* such that $p(x_*) = 0$. Now suppose there are two such points x_1 and x_2 such that $2/3 < x_1 < x_2 < 1$ (W.L.O.G). Then $p(x_1) < p(x_2)$ because $p'(x) = 3x^2 + 2x = x(3x + 2) > 0$ for $x \in (2/3, 1)$, but $p(x_1) = p(x_2) = 0$. This forms a contradiction, and hence there is only one such point. $\square$