The book over-complicates this chapter in my opinion, so I will compress it and simplify it for the purposes of this class. Instead of discussing induction as theorems and proving it works, let us take for granted that it works because it is fairly intuitive, and spend that time in working on examples instead.

Property 1 (Well-ordering of $\mathbb{N}$)**.** Every nonempty subset of $\mathbb{N}$ has a least element.

For example $\{5, 6, 25, 30\}$ has a least element of 5. Since this can be done for all subsets in $\mathbb{N}$, the elements of $\mathbb{N}$ will be ordered. Further, notice that this can be done for similar sets, for example, the integers.

Property 2 (Weak induction)**.** Suppose the following statements hold
(1) $P(1)$ is true, and
(2) For all $k > 1$, $P(k) \Rightarrow P(k+1)$, then
$P(n)$ is true for all $n \in \mathbb{N}$.

This may seem complicated, but it boils down to three steps for induction proofs:
(1) Base case: Show that the first case ($P(1)$) is true.
(2) Inductive hypothesis: Assume that the "current" case $P(k)$ is true.
(3) Inductive step: Prove that the "future" case $P(k+1)$ is true.

Lets look at a simple example of this.

Ex: $\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)$.

Proof. Base case: $\sum_{n=1}^{1} n = 1$.
Hypothesis: $\sum_{n=1}^{k} \frac{1}{2}k(k+1)$.
Induction: Using the hypothesis and the property of sums we get

$$\sum_{n=1}^{k+1} n = \sum_{n=1}^{k} n + (k+1) = \frac{1}{2}k(k+1) + (k+1) = (k+1)(\frac{1}{2}k+1) = \frac{1}{2}(k+1)(k+2),$$

which is precisely what we would get if we had plugged into the formula directly. □

Often we may need to use past cases in addition to the current case.

Property 3 (Strong induction)**.** Suppose the following statements hold
(1) $P(1)$ is true, and
(2) For all $k > 1$, $P(1), P(2), \ldots, P(k) \Rightarrow P(k+1)$, then
$P(n)$ is true for all $n \in \mathbb{N}$.

Now lets look at a really difficult example that is like what you would see in Advanced Calculus.

Example 1. *The n^{th} term of the Fibonacci sequence ($\{1, 1, 2, 3, 5, 8, \ldots, n+1 = n + (n-1)\}$) is given by the following formula,*

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{1}$$

Proof. Base case: $F_1 = 1$, $F_2 = 1$ ✓
Hypothesis: Suppose F_k and F_{k-1} hold, then
Inductive step:

$$F_{k+1} = F_k + F_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right].$$

Notice that

$$\frac{1+\sqrt{5}}{2}+1=\frac{3+\sqrt{5}}{2}=\left(\frac{1+\sqrt{5}}{2}\right)^2,$$

and

$$\frac{1-\sqrt{5}}{2}+1=\frac{3-\sqrt{5}}{2}=\left(\frac{1-\sqrt{5}}{2}\right)^2,$$

hence

$$F_{k+1}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Therefore, the formula (1) is true.

□

Now lets look at some book problems.

6.4) *Proof.* Base case: $1 = 1^2$ ✓

Hypothesis: $\sum_{n=1}^{k} 2n-1 = k^2$.

Inductive step:

$$\sum_{n=1}^{k+1} 2n-1=\sum_{n=1}^{k} 2n-1+(2k+1)=k^2+2k+1=(k+1)^2,$$

which is precisely what we would get if we had plugged directly into the formula. □

6.18) *Proof.* Base case: $2^10 = 1024 > 10^3 = 1000$ ✓

Hypothesis: $2^k > k^3$

Inductive step: We would like to show that if $2^k > k^3$, then $2^{k+1} > (k+1)^3$.

Notice that $2^k > k^3 \Rightarrow 2^{k+1} > 2k^3$. Further, $(k+1)^3 = k^3+3k^2+3k+1$. Since $k^3 > 7k^2 > 3k^2+3k^2+k^2 > 3k^2+3k+1$ when $k > 7$,

$$2^{k+1} > 2k^3 > k^3+7k^2 > k^3+3k^2+3k+1=(k+1)^3.$$

□

6.33) Since this is not a discrete math course I will provide the formula. Here it is $a_n = 2^{n-1}$ for $n \geq 1$.

Proof. Base case: $a_1 = 1$ ✓

Hypothesis: $a_k = 2^{k-1}$, $a_{k-1} = 2^{k-1}$, $\ldots$, $a_1 = 1$.

Inductive step: $a_{k+1} = 2a_k = 2 \cdot 2^{k-1} = 2^k$, which is precisely what we would get if we plugged in directly to the formula. □