Most important things to know: induction, proving limits, using limit laws.

Definition 1. A sequence $\{x_n\} \subseteq \mathbb{R}$ converges if there is an $x \in \mathbb{R}$ such that
For every $\varepsilon > 0$, there is an $\mathbb{N}$ such that $|x - x_n| \leq \varepsilon$ for all $n \geq \mathbb{N}$;
otherwise it diverges. We call this x the limit of $\{x_n\}$.

Theorem 1. *If $x_n \to p$, then $\{x_n\}$ is bounded.*

Theorem 2 (Squeeze). *For $\{x_n\}, \{y_n\}, \{z_n\} \subseteq \mathbb{R}$, if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then $\{y_n\}$ is convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$.*

Theorem 3. *If $x_n \to p$, $|x_n| \to |p|$.*

Theorem 4 (Monotone Convergence). *A monotone sequence converges if and only if it is bounded, and if $\{x_n\}$ is bounded and increasing,*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}, \tag{1}$$

and if $\{y_n\}$ is bounded and decreasing,

$$\lim_{n\to\infty} y_n = \inf\{y_n : n \in \mathbb{N}\} \tag{2}$$

Property 1 (Divergence criteria). If $\{x_n\} \subseteq \mathbb{R}$, then it diverges if
(1) it is unbounded, or
(2) it has convergent subsequences with differing limits.

Lemma 1 (Monotone subsequences). *If $\{y_n\} \subseteq \mathbb{R}$, it has a monotone subsequence.*

Theorem 5 (Bolzano–Weierstrass). *If $\{x_n\} \subseteq \mathbb{R}$ is bounded, it has a convergent subsequence.*

Lemma 2. *If $\{x_n\} \subseteq \mathbb{R}$ converges, then for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $n, m \geq N$.*

Definition 2. A sequence $\{x_n\} \subseteq \mathbb{R}$ is called a Cauchy sequence if
for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $n, m \geq N$.
And this property is called the Cauchy criterion.

Theorem 6 (Cauchy sequences). *In $\mathbb{R}$ every Cauchy sequence converges.*

Induction: For induction make sure you prove the base case first, then make the inductive assumption, then prove the inductive step. It is not enough to show it for a few cases, nor is it enough to just show the inductive step (until after you graduate that is).

Proving limits: To prove the existence of a limit, first do a little scratch work to see what $N(\varepsilon)$ has to be to make your entire quantity less than ε, but be careful to make sure that N is increasing as ε is decreasing, otherwise you picked the wrong N.

Definition 3. Let $x_0, \varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. Then the ε-neighborhood (ball) around x_0 is
$B_\varepsilon(x_0) := \{x \in \mathbb{R} : |x - x_0| < \varepsilon\}$.

Notice that in the real line, a neighborhood or ball, is just an open interval, so $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$.

Most important homework problems: Compulsory: Hw2 # 1; Hw3 # 3-6; Hw4 All;
Supplementary: 1.2 # 1,6; 2.1 # 26; 2.2 # 16; 3.1 # 5; 3.2 # 20, 23; 3.3 # 1-3; 3.4 # 3, 9.
Additional problems you can look at: 1.2 # 2 - 5, 7 - 9, 13 - 15, 18; 3.1 # 4, 6, 16, 17; 3.2 # 1, 5, 6, 9 - 17; 3.3 # 4 - 7, 11, 12; 3.4 # 7, 8.

The exam is organized as follows: 45 points will be mainly calculations with the use of a little bit of theory (e.g. proving a limit), 45 points will be using theorems/logic to prove the results, and 10 points will be a completely theoretical problem involving sets, topology, and sequences all in one.

Note: I will never ask you to prove a theorem. I don't want you memorizing proofs!