5.1 CONTINUITY

Definition 1. The function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$;
i.e., for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all x $0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$;
otherwise f is said to be discontinuous at $x = a$.

Remark: It should be noted that we can write $f(B_\delta(c)) \subseteq B_\varepsilon(f(c))$.

Definition 2. Let $A \subset \mathbb{R}$ and let $f : A \mapsto \mathbb{R}$. If $B \subseteq A$, we say that f is continuous on B if f is continuous at every point in B.

Notice that the format of proving a specific function is continuous is the same as proving a limit exists. Now lets look at some examples from the book.

4.1.12 d) In this problem they want us to prove that the limit $\lim_{x\to 0} \sin \frac{1}{x^2}$ does not exist. So we need to show that for all δ and some x there is an ε such that $|x - 0| = |x| < \delta \Rightarrow |f(x) - L| < \varepsilon$ for all L. Let us prove this by taking a few cases of L.

Proof. Consider $|L| > 1$, then $|\sin(1/x^2) - L| \geq |1 - L|$, so choose $\varepsilon = |1 - L|$. If $|L| = 1$, there is an x such that $\sin(1/x^2) = 0$, then choose $\varepsilon = 1$. If $|L| < 1$, there is an x such that $\sin(1/x^2) = 1$, then choose $\varepsilon = (1 - L)/2$. □

5.1.1) This wants us to prove the following theorem

Theorem 1. *A function $f : A \mapsto \mathbb{R}$ is continuous at $c \in A$ if and only if for all sequences $\{x_n\} \subseteq A$, $x_n \to c \Rightarrow f(x_n) \to f(c)$.*

Proof. Suppose $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$, so if there is an N such that $|x_n - c| < \delta$ for all $n \geq N$, then $|f(x_n)-f(c)| < \varepsilon$ for all $n \geq N$, so $x_n \to c$. Suppose there is an N such that for all $n \geq N$ $|x_n-c| < \delta \Rightarrow |f(x_n)-f(c)| < \varepsilon$. Then there is a $\delta > 0$ such that $|x_n-c| < \delta \Rightarrow |f(x_n)-f(c)| < \varepsilon$. □

5.1.2) This wants us to prove the following theorem

Theorem 2. *Let $A \subseteq \mathbb{R}$, $f : A \mapsto \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to c$, but $f(x_n) \not\to f(c)$.*

Proof. This is the contrapositive of the previous theorem. □

5.1.10) $\lim_{x\to c} |x| = |c|$.

Proof. If $|x - c| < \delta$, by the triangle inequality,

$$||x| - |c|| \leq ||x - c|| = |x - c| < \delta,$$

so, choose $\delta = \varepsilon$. □

5.1.11) For this one lets write it as a definition and theorem.

Definition 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous if $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$, where $K > 0$.

Theorem 3. *If f is Lipschitz it is continuous for all $c \in \mathbb{R}$*

Remark: In fact we will show later that it is uniformly continuous.

Proof. Suppose $|x-y| < \delta$, then $|f(x)-f(y)| < K\delta$. If $y = c$, $|x-y| < \delta \Rightarrow |f(x)-f(c)| \leq K\delta < 2K\delta$. Then choose $\delta = \varepsilon/2K$. □