6.1 The Derivative

While we may define this using the formal definition of a limit, it is perhaps more instructive to use our new machinery in a more straightforward way.

Definition 1. The function <u>f is differentiable at $x = a$</u> if

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

exists. The limit is denoted by $f'(a)$ and is called the <u>derivative</u>.

Notice that this definition is written in a slightly different way, but you should convince yourself that these two definitions are equivalent.

Now lets look at a bunch of simple examples applying the definition

Ex: $f(x) = c$.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0.$$

This means that f is differentiable at every point $x = a$ and $f'(a) = 0$.

Ex: $f(x) = cx + d$.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{c\cdot(a+h) + d - [ca+d]}{h} = \lim_{h\to 0} \frac{ch}{h} = c$$

So for all points $x = a$, $f'(a) = c$.

Ex: $f(x) = x^2$.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} 2a + h = 2a.$$

Ex: $f(x) = x^3$.

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} = \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3}{h} = \lim_{h\to 0} 3a^2 + 3ah + h^2 = 3a^2.$$

Ex: $f(x) = |x|$. Lets see if this is differentiable at $x = 0$ (obviously it's not)

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

Then for $h \geq 0$, $|h|/h = 1$ and for $h < 0$, $|h|/h = -1$. So the derivative does not exist. However we can define right and left hand derivatives from the right and left hand limits. For this functions these do exist independently.

$$\text{RHD:} \quad \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$\text{LHD:} \quad \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}$$

Ex: $f(x) = \sqrt{x}$. Lets see if this is differentiable at $x = 0$,

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h}} = \infty.$$

Here there is no left hand derivative, and the right hand derivative blows up, so the derivative does not exist.

In order for a derivative to exist, both the left and right hand limits must be equal.

Now lets look at what differentiability says about continuity.

Theorem 1. *If f is differentiable at $x = a$ then it is continuous at $x = a$.*

Proof. Consider the following limit,

$$\lim_{h \to 0} f(a+h) - f(a). \tag{2}$$

If it is zero then we will have proved continuity since by definition continuity is $\lim_{x \to a} f(x) = f(a)$. We write

$$\lim_{h \to 0} f(a+h) - f(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot h = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h.$$

We can do this because f is differentiable at $x = a$, and hence the limit exists. Also, $\lim_{h \to 0} h = 0$, so we have

$$\lim_{h \to 0} f(a+h) - f(a) = 0 \Rightarrow \lim_{h \to 0} f(a+h) = f(a),$$

which is precisely the definition of continuity. □

I will skip a bunch of stuff because you would have seen it in Calc I, but make sure you read through 6.1.3 - 6.1.8. Now lets look at something you most likely haven't seen before.

Theorem 2. *Let f be continuous and one-to-one on an interval and suppose f is differentiable at $f^{-1}(b)$ such that $f'(f^{-1}(b)) \neq 0$. Then f^{-1} is differentiable at $x = b$, and*

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}. \tag{3}$$

Proof. Let $b = f(a)$. Then,

$$f^{-1}(b) = \lim_{h \to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{h \to 0} \frac{f^{-1}(b+h) - a}{h}.$$

We may write $b + h = f(a + k(h))$ as long as $b + h$ is in the domain of f^{-1}. Then

$$\lim_{h \to 0} \frac{f^{-1}(b+h) - a}{h} = \lim_{h \to 0} \frac{f^{-1}(f(a+k)) - a}{f(a+k) - b} = \lim_{h \to 0} \frac{k}{f(a+k) - f(a)}.$$

From a previous theorem f^{-1} is continuous at b, so $k \to 0$ as $h \to 0$ since $k = f^{-1}(b+h) - f^{-1}(b)$. Therefore,

$$\lim_{k \to 0} \frac{f(a+k) - f(a)}{k} = f'(a) = f'(f^{-1}(b)) \neq 0 \Rightarrow (f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

□