2.1 Properties of $\mathbb{R}$

We listed properties 2.1.1 - 2.1.3 from the book.

Recall that we can write even and odd numbers as $n = 2k$ and $n = 2k + 1$. Notice that $(2k)^2 = 4k^2 = 2 \cdot (2k)^2$ and $(2k+1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1$. So, the square of evens are even and the square of odds are odd. Further, notice that by the fundamental theorem of arithmetic if x^2 is divisible by p where p is a prime, then so is x. We can use these properties to prove that certain square roots are irrational.

Theorem 1. $\sqrt{3}$ *is irrational.*

Proof. Suppose not; i.e. $\sqrt{3}$ is rational. Then $\sqrt{3} = m/n$ such that $m, n \in \mathbb{N}$ and suppose this is in lowest form (since we can factor); i.e. m and n have no common factors. Then $m^2 = 3n^2$, so m is divisible by 3; i.e. $m = 3k \Rightarrow 9k^2 = 3n^2 \Rightarrow n^2 = 3k^2$, so n is divisible by 3, but m and n have no common factors. This forms a contradiction, therefore $\sqrt{3}$ is not rational. $\square$

Next we briefly went over 2.1.5 - 2.1.8 before proving the next theorem

Theorem 2. *If $x \in \mathbb{R}$ such that $0 \le x < \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Proof. Suppose $x > 0$, then we can chose $\epsilon = x/2$, which forms a contradiction. Therefore, $x = 0$. $\square$

Then we finished off the section by going over 2.1.10 and 2.1.11.

2.2 Absolute value and $\mathbb{R}$

We first went over 2.2.1 and 2.2.2 before showing the triangle inequality.

Theorem 3 (Triangle Inequality)**.** *For $x, y \in \mathbb{R}$, $|x + y| \le |x| + |y|$.*

Proof. Since $-|x| \le x \le |x|$ and $-|y| \le y \le |y|$, $-(|x| + |y|) \le x + y \le |x| = |y|$, then $|x + y| \le |x| + |y|$. $\square$

What about for $|x| - |y|$? Notice that $|x| \ge |x + y| - |y|$. If we let $x = a + b$ and $y = -b$, then $|a + b| \ge |a| - |b|$. Also, if $x = a - b$ and $y = b$, then $|a - b| \ge |a| - |b|$. It's also easy to show that the triangle inequality holds for more than two elements by applying the original triangle inequality twice: $|x + y + z| \le |x| + |z + y| \le |x| + |y| + |z|$.

Definition 1. Let $x_0, \epsilon \in \mathbb{R}$ such that $\epsilon > 0$. Then the ϵ-<u>neighborhood (ball)</u> around x_0 is $B_\epsilon(x_0) := \{x \in \mathbb{R} : |x - x_0| < \epsilon\}$.

Theorem 4. *If $x \in B_\epsilon(x_0)$ for all $\epsilon > 0$, then $x = x_0$.*

Proof. Notice that $|x - x_0| < \epsilon$ for all $\epsilon > 0$, then $x = x_0$. $\square$

This proof is a direct application of Theorem 2 from the previous section. In essence this says that x_0 is the only element contained in every ball about itself.

2.3 COMPLETENESS OF $\mathbb{R}$

First we define some preliminaries.

Definition 2. Let $S \subseteq \mathbb{R}$ such that $S \neq \emptyset$, then
   (1) S is <u>bounded above</u> if there is an $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$, where each M is called an <u>upper bound</u> of S.
   (2) Similarly S is <u>bounded below</u> if there is an $m \in \mathbb{R}$ such that $s \geq m$ for all $s \in S$, where each m is called a <u>lower bound</u> of S.
   (3) A set is said to be <u>bounded</u> if it is bounded from both above and below, otherwise it is said to be <u>unbounded</u>.

Definition 3. With the same general assumptions as above,
   (1) If S is bounded above, M is a <u>supremum (sup S)</u> if
      (a) M is an upper bound of S, and
      (b) $M \leq M_1$ for all upper bounds M_1 of S.
   (2) If S is bounded below, m is an <u>infimum (inf S)</u> if
      (a) m is a lower bound of S, and
      (b) $m \geq m_1$ for all lower bounds m_1 of S.

For example $\sup[0,1] = \sup[0,1) = 1$. So, the supremum can be in the set or out of the set. There are also many other upper and lower bounds of these sets, but the sup and inf are special.

Axiom 1 (Completeness of $\mathbb{R}$)**.** Every nonempty set $S \subseteq \mathbb{R}$ that is bounded above has a supremum; i.e. there is an $M \in \mathbb{R}$ such that $M = \sup S$.

For example, $\mathbb{R} \setminus \{0\}$ is not complete. Consider $S = [-1, 0)$. Can we find a supremum? We cannot since $\sup S = 0$ in $\mathbb{R}$, but $\{0\}$ is not part of our new set.

Theorem 5 (Approximation property)**.** *Let $S \subseteq \mathbb{R}$ such that $S \neq \emptyset$ with $M = \sup S$. Then for all $a < M$ there is an $x \in S$ such that $a < x \leq M$.*

Proof. Notice $x \leq M$ for all $x \in S$. If $x \leq a$ for all $x \in S$, then a would be an upper bound smaller than the supremum, which forms a contradiction. Therefore, there is at least one $x \in S$ where $x > a$. □

Can $\inf S = \sup S$? It can if there is only one element in the set S. This is because $\inf S \leq x \leq \sup S$ for all $x \in S$, and since $\inf S = \sup S$, $\inf S = x = \sup S$ for all $x \in S$, which means $S = \{x\}$. We can also do this proof using balls, but that would not be the "book proof".

2.4 APPLICATIONS OF THE SUPREMUM

Theorem 6 (Additive property)**.** *Given nonempty subsets $A, B \subseteq \mathbb{R}$, let $C = \{x + y : x \in A, y \in B\}$. If A and B have supremums, then so does C and $\sup C = \sup A + \sup B$.*

Proof. If $z \in C$ then there is an $x \in A$ and $y \in b$ such that $z = x + y$, so $z \leq \sup A + \sup B \Rightarrow \sup A + \sup B$ is an upper bound of C. Hence C has a supremum (by completeness) and $\sup C \leq \sup A + \sup B$.

Next we must show $\sup A + \sup B \leq \sup C$. For $\epsilon > 0$, there is an $x \in A, y \in B$ such that $\sup A - \epsilon < x$ and $\sup B - \epsilon < y$. Then adding the two gives us $\sup A + \sup B - 2\epsilon < x + y \leq z \leq \sup C$. Therefore, $\sup A + \sup B < \sup C + 2\epsilon$ for all ϵ, so by Theorem 2, $\sup A + \sup B \leq \sup C$. Thereby completing the proof. □

Theorem 7 (Comparison property)**.** *Given nonempty sets $S, T \subseteq \mathbb{R}$ such that $s \leq t$ for all $s \in S, t \in T$. If T has a supremum, so does S and $\sup S \leq \sup T$.*

Proof. $s \leq t \leq \sup T$ means that S has an upper bound, and hence by completeness it has a supremum. If $\sup S > \sup T$, then there would exist s, t such that $s > t$, which forms a contradiction. Therefore, $\sup S \leq \sup T$. □

Notice that if $\sup S > \sup T$, $s > t$ if we chose $s = \frac{1}{2}(\sup S + \sup T)$.

Theorem 8. $\mathbb{Z}^+$ *is unbounded above.*

Proof. Suppose $\mathbb{Z}^+$ is bounded, then it has a supremum. Then there is an $n \in \mathbb{Z}^+$ such that $\sup \mathbb{Z}^+ - 1 < n$ (we use -1 since the smallest $\epsilon \in \mathbb{Z}$ such that $\epsilon > 0$ is $\epsilon = 1$). Since $n + 1 \in \mathbb{Z}^+$ and $n + 1 > \sup \mathbb{Z}^+$, this forms a contradiction, thereby completing the proof. $\square$

Note that Theorem 2.4.3 in the book follows directly from this.

Also, in my opinion the density theorem theorem in the book does not give a complete picture, so lets discuss it after intervals. However, make sure you go through the proofs in the book.