## 3.2 Limit Theorems

Here we'll just prove a bunch of theorems.

**Definition 1.** A sequence $\{x_n\} \subseteq \mathbb{R}$ is bounded if there is a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

**Theorem 1.** *If $x_n \to p$, then $\{x_n\}$ is bounded.*

*Proof.* Let $N(\epsilon) \in \mathbb{Z}$ such that $\epsilon = 1$. Since $x_n \to p$, for all $n \geq N$, $x_n \in B_1(p)$, so all $x_N \in B_r(p)$ where $r = 1 + \max\{|p - x_1|, |p - x_2|, \dots, |p - x_{n-1}|\}$. □

Notice, that after we took care of the infinite part of the sequence, we can take care of the finite part one by one since it is countable. And of course, a set of finite points can never be unbounded.

Theorems 3.2.4 - 3.2.6 basically say, If $x_n \to p$ and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq p \leq b$.

**Theorem 2** (Squeeze)**.** *For $\{x_n\}, \{y_n\}, \{z_n\} \subseteq \mathbb{R}$, if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then $\{y_n\}$ is convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$.*

*Proof.* Let $x_n \to p$, $z_n \to p$. For $\epsilon > 0$ there is $n, N \in \mathbb{N}$ such that if $n \geq N$, $|x_n - p| < \epsilon$ and $|z_n - p| < \epsilon$. Since $x_n \leq y_n \leq z_n$, $x_n - p \leq y_n - p \leq z_n - p$, and hence $-\epsilon \leq y_n - p \leq \epsilon$ for all $n \geq N, \epsilon$. Therefore, $|y_n - p| \to 0 \Rightarrow y_n \to p$. □

**Theorem 3.** *If $x_n \to p$, $|x_n| \to |p|$.*

*Proof.* By the triangle inequality, $||x_n| - |p|| \leq |x_n - p| \to 0$ because $x_n \to p$, therefore $|x_n| \to |p|$. □

**Theorem 4.** *Let $\{x_n\} \subseteq \mathbb{R}^+$ such that $\frac{x_{n+1}}{x_n} \to L$. If $L < 1$, $x_n \to 0$.*

*Proof.* Let $L < r < 1$, and $\epsilon = r - L$. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\frac{x_{n+1}}{x_n} - L| < \epsilon$. Then by the triangle inequality and since we are in the positive reals,

$$\frac{x_{n+1}}{x_n} - L = \left|\frac{x_{n+1}}{x_n}\right| - |L| \leq \left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon \Rightarrow \frac{x_{n+1}}{x_n} < L + \epsilon = r.$$

So, $0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_N r^{n-N+1}$. Let $C = x_N / r^N$ because this is the finite part of our expression; i.e. $N$ is not going off to infinity. Then, $0 < x_{n+1} < C r^{n+1}$. Since $0 < r < 1$, $r^n \to 0$, therefore $x_n \to 0$. □

## 3.3 Monotone Sequences

In this section we shall see how much easier convergence proofs get if we have the additional property of monotonicity.

**Definition 2.** Let $\{x_n\} \subseteq \mathbb{R}$. It is increasing if $x_n \leq x_{n+1}$ and decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If it is either decreasing or increasing, it is said to be monotone.

**Theorem 5** (Monotone Convergence)**.** *A monotone sequence converges if and only if it is bounded, and if $\{x_n\}$ is bounded and increasing,*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}, \tag{1}$$

*and if $\{y_n\}$ is bounded and decreasing,*

$$\lim_{n\to\infty} y_n = \inf\{y_n : n \in \mathbb{N}\} \tag{2}$$

We showed that a convergent sequence is bounded, but is a bounded sequence necessarily convergent? In class we looked at the example $x_n = (-1)^n$. We can also think of the example

$$x_n = \begin{cases} 0 & n \text{ odd,} \\ 1 & n \text{ even;} \end{cases} \tag{3}$$

*Proof.* We showed that a convergent sequence is bounded in Theorem 3.2.2.

Now let $\{x_n\}$ be a bounded monotone sequence. Without loss of generality, assume that it is increasing. Since it is bounded, by completeness, it has a supremum, which we will call $x_*$. Since this is a supremum, there is an $x_N$ such that $x_* - \epsilon < x_N$ for all $\epsilon > 0$. By the definition of increasing, $x_N \leq x_n$ for all $n \geq N$, so

$$x_* - \epsilon < x_N \leq x_n < x_* + \epsilon \Rightarrow |x_n - x_*| < \epsilon$$

for all $n \geq N$, and hence $x_n \to x_*$. □

3.1.11 The book asks us to prove $1/(n^2+n)$ is decreasing and bounded below. Lets actually prove that it is convergent though.

*Proof.* The sequence is bounded since $\frac{1}{n^2+n} \geq 0$ because $n > 0$. Furthermore, by induction (the proof of which is left to the reader)

$$\frac{1}{(n+1)^2+(n+1)} = \frac{1}{n^2+3n+2} = \frac{1}{(n^2+n)+(2n+2)} \leq \frac{1}{n^2+n},$$

because $2n+2 > 0$. Since the sequence is bounded below and decreasing, it converges. Moreover, it converges to zero since that is the infimum of the sequence (the proof of which is again left to the reader). □

3.1.15 The book asks us to prove $(2n)^{1/n}$ is not monotone.

**Solution:** Notice that while it initially increases: $2, 2, 9^{1/3}$, since its limit is unity, it eventually decreases. There is merit to this sort of sequence, however, since it is eventually monotone.

Ex: Notice that $r^n$ is increasing for $-1 < r < 0$ and decreasing for $0 < r < 1$.