3.5 Cauchy Sequences Continued

An interesting type of Cauchy sequence is a contractive sequence.

Definition 1. A sequence $\{x_n\}$ is contractive if there is $0 < C < 1$ such that $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

Theorem 1. *Every contractive sequence is Cauchy.*

Proof. Notice that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \leq C^2|x_n - x_{n-1}| \leq \cdots \leq C^n|x_2 - x_1|$$

Then, without less of generality, if $m > n$,

$$\begin{aligned}|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| &\leq \left(C^{m-2} + C^{m-3} + \cdots + C^{n-1}\right)|x_2 - x_1| \\ &= C^{n-1}\left(\frac{1 - C^{m-n}}{1 - C}\right)|x_2 - x_1| \leq C^{n-1}\left(\frac{1}{1 - C}\right)|x_2 - x_1|.\end{aligned}$$

Since $0 < C < 1$, and $1/(1 - C)$ is constant, choose

$$N(\varepsilon) > \frac{\ln \frac{(1-C)\varepsilon}{|x_2 - x_1|}}{\ln C} + 1,$$

then $|x_m - x_n| < \varepsilon$ for all $n \geq N$. □

3.7 Infinite Series

We know what series are from Calc II, so lets jump right into it

Theorem 2 (n^{th} term test)**.** *If $\sum x_n$ converges, then $x_n \to 0$.*

Proof. Consider the partial sum $S_N = \sum_{n=1}^{\infty} x_n$. If $\sum x_n$ converges then the sequence of partial sums S_N must also converge, say to S. Then for all $\varepsilon > 0$, $|x_n| = |S_N - S_{N-1}| < \varepsilon$ for all $N \geq M$ because a convergent sequence is also Cauchy. Since this holds for all ε, $x_n \to 0$. □

Theorem 3 (Direct Comparison)**.** *If $0 \leq x_n \leq y_n$ for all $n \geq K \in \mathbb{N}$, then*

(1) *If $\sum y_n$ converges so does $\sum x_n$.*
(2) *If $\sum x_n$ diverges so does $\sum y_n$.*

Proof. If $\sum y_n$ converges, then clearly $S_N = \sum_{n=K}^{N} y_n$ converges, which means that it is Cauchy. And since $x_n \leq y_n$ for all $n \geq K$, $R_N = \sum_{n=K}^{N} \leq S_N$. Further, for all $\varepsilon > 0$, if we assume without loss of generality that $M > N$, we have

$$|R_M - R_N| = |x_M + x_{M-1} + x_{M-2} + \cdots + x_{N+1}| \leq |y_M + y_{M-1} + y_{M-2} + \cdots + y_{N+1}| = |S_M - S_N| < \varepsilon$$

for all $M, N \geq K$. □

Theorem 4 (Limit Comparison)**.** *Suppose $x_n, y_n > 0$ and $\frac{x_n}{y_n} \to L$, then*

(1) *If $L \neq 0$, then $\sum x_n$ and $\sum y_n$ both converge or both diverge, and*
(2) *if $L = 0$, if $\sum y_n$ converges, so does $\sum x_n$.*

Proof. (1) Since $\frac{x_n}{y_n} \to L \neq 0$, there is an $N \in \mathbb{N}$ such that

$$\frac{1}{2}L \leq \frac{x_n}{y_n} \leq 2L \Rightarrow \frac{1}{2}Ly_n \leq x_n \leq 2Ly_n.$$

So, by direct comparison if without loss of generality $\sum y_n$ converges, then so does $\sum x_n$.

(2) If $L = 0$, then $0 \leq x_n \leq y_n$, and again by direct comparison if $\sum y_n$ converges so does $\sum x_n$ □

What is a direct consequence of part b?