4.2 Continued

Here is a theorem that we saw and proved for sequences.

Theorem 1 (Squeeze)**.** *Let $f, g, h : A \to \mathbb{R}$ and let $f(x) \le g(x) \le h(x)$. If $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$.*

We won't prove it, but the proof is similar to that of sequences.
Here is another useful theorem about being bounded away from zero.

Theorem 2. *Let $f : A \to \mathbb{R}$. If $\lim_{x\to a} f(x) > 0$, then for $(a-\delta, a+\delta)$, $f(x) > 0$ for some $\delta > 0$ and all $x \in A \cap (a-\delta, a+\delta) \setminus \{a\}$.*

And here is a theorem about how we can relate the limit of functions to that of sequences.

Theorem 3.

Now lets work through a bunch of examples.

Ex: $\lim_{x\to 0} x \sin \frac{1}{x} = 0$.

This is an easy one so we don't have to do any scratch work. We just have to notice that sine is bounded by 1 in absolute value.

Proof. Choose $\delta = \varepsilon$, then for all $\varepsilon > 0$, we have

$$\left| x \sin \frac{1}{x} \right| \le |x| < \delta = \varepsilon.$$

□

4.1.10a) $\lim_{x\to 2}(x^2 + 4x) = 12$.

For this one we have to do a bit more work. What we have is $|x-2| < \delta$, and what we would like to show is $|x^2 + 4 - 12| < \varepsilon$. Lets try to factor this out and isolate $|x-2|$,

$$|x^2 + 4x - 12| = |x-2||x+6| < 9|x-2|$$

if $|x-2| < 1$; i.e., if $\delta = 1$. Further, since $|x-2| < \delta$ and we want $9|x-2| < \varepsilon$, another choice of δ is $\varepsilon/9$.

Proof. Choose $\delta = \min(1, \varepsilon/9)$, then for all $\varepsilon > 0$, we have

$$|x^2 + 4x - 12| = |x-2||x+6| < 9|x-2| < \varepsilon.$$

□

4.1.10b) $\lim_{x\to -1} \frac{x+5}{2x+3} = 4$.

This takes even more work, but the ideas are the same. The first thing we notice is that $x \ne -3/2$, so we need to keep this in mind for our choice of δ. Before we go to make any choices lets see what happens after factoring

$$\left| \frac{x+5}{2x+3} - \frac{8x+12}{2x+3} \right| = \left| \frac{-7x-7}{2x+3} \right| = \left| \frac{7}{2x+3} \right| |x+1|.$$

Now, we already know that $|x+1| < \delta$, so all we need to do is bound the term in front of it. Obviously because $x \ne -3/2$, $\delta < 1/2$, so lets pick $\delta = 1/4$ for the bounding. Notice that this is not **THE** choice of δ, but rather **A** choice we make purely for the sake of bounding. Now we can pick any δ between 1/2 and 0, so lets pick 1/4. Notice that since we have x in the denominator, the infimum of x will determine the supremum of $7/2x + 3$, and $\inf(x) = -5/4$. Then we can bound and relate δ and ε

$$\left| \frac{7}{2x+3} \right| |x+1| < 14\delta = \varepsilon.$$

Now we are ready to write our proof.

Proof. Choose $\delta = \min(1/4, \varepsilon/14)$, then for all $\varepsilon > 0$, we have

$$\left|\frac{x+5}{2x+3} - \frac{8x+12}{2x+3}\right| = \left|\frac{-7x-7}{2x+3}\right| = \left|\frac{7}{2x+3}\right||x+1| < \varepsilon.$$

□

4.1.9b) $\lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}$.

This one is pretty easy. We do have to stay away from $x = -1$, but that shouldn't be too hard. Lets factor first

$$\left|\frac{x}{1+x} - \frac{1}{2}\right| = \left|\frac{2x-1-x}{2+2x}\right| = \left|\frac{1}{2+2x}\right||x-1|.$$

Then again since x is in the denominator we look to find the least x in order to maximize $|1/2+2x|$. Notice that here we are safe to choose $\delta = 1$, so lets do that. Then $|1/(2+2x)| < 1/2$ since $\inf(x) = 1/2$. Now we can write our proof.

Proof. Choose $\delta = \min(1, 2\varepsilon)$, then for all $\varepsilon > 0$,

$$\left|\frac{x}{1+x} - \frac{1}{2}\right| = \left|\frac{2x-1-x}{2+2x}\right| = \left|\frac{1}{2+2x}\right||x-1| < \frac{\delta}{2} = \varepsilon.$$

□

4.1.12a) $\lim_{x\to 0} 1/x^2$ does not exist.

Proof: $\varepsilon - \delta$. If $L \leq 0$, $|1/x^2 - L| \geq |1/x^2| > 1/\delta^2$. Choose $\varepsilon = 1/\delta^2$.

If $L > 0$, suppose $|x| < \delta$, then choose $x = 1/L$. If $\delta > 1/L$, choose $\varepsilon = L^2 - L$. If $\delta \leq L$, choose $\varepsilon = 1/\delta^2 - L$. □

Proof: Sequences. Choose $x = 1/n$, then $1/x^2 = n^2$. We know $\{n^2\}$ diverges since for all $n \geq N$, $n^2 \geq N^2$; i.e., it is not bounded. □

4.3 Infinite limits

Here we will just go over a bunch of definitions and theorems.

Definition 1. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $a \in \mathbb{R}$ be a limit point of A. Then

(1) We say $\underline{f \to \infty \text{ as } x \to a}$; i.e., $\lim_{x\to a} f(x) = \infty$, if for all $M \in \mathbb{R}$ there is a $\delta(M) > 0$ such that for all $x \in A$, $0 < |x-a| < \delta \Rightarrow f(x) > M$.

(2) We say $\underline{f \to -\infty \text{ as } x \to a}$; i.e., $\lim_{x\to a} f(x) = -\infty$, if for all $m \in \mathbb{R}$ there is a $\delta(m) > 0$ such that for all $x \in A$, $0 < |x-a| < \delta \Rightarrow f(x) > m$.

Definition 2. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say $L \in \mathbb{R}$ is a $\underline{\text{limit of } f \text{ as } x \to \infty}$; i.e., $\lim_{x\to\infty} f(x) = L$, if for all $\varepsilon > 0$ there is a $K(\varepsilon) > a$ such that for all $x > K$, $|f(x) - L| < \varepsilon$.

Theorem 4. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $a \in \mathbb{R}$ be a limit point of A. Suppose $f(x) \leq g(x)$ for all $x \in A$, $x \neq a$. Then*

(1) $\lim_{x\to a} f(x) = \infty \Rightarrow \lim_{x\to a} g(x) = \infty$

(2) $\lim_{x\to a} g(x) = -\infty \Rightarrow \lim_{x\to a} f(x) = -\infty$

Proof. By definition for all $M \in \mathbb{R}$ there is a $\delta(M) > 0$ such that $|x-a| < \delta \Rightarrow f(x) > M$ for all $x \in A$. And since $g(x) > f(x) > M$, $\lim_{x\to a} g(x) = \infty$. □

Theorem 5. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose $g(x) > 0$ for all $x > a$ and $\lim_{x\to\infty} f(x)/g(x) = L$. Then*

(1) *If $L > 0$, $\lim_{x\to a} f(x) = \infty \Rightarrow \lim_{x\to a} g(x) = \infty$*

(2) *If $L < 0$, $\lim_{x\to a} f(x) = -\infty \Rightarrow \lim_{x\to a} g(x) = \infty$*

Proof. Since $L > 0$, there is an $a_1 > a$ such that $0 < L/2 \leq f(x)/g(x) < 3L/2$ for $x > a_1$. This means

$$\frac{1}{2}Lg(x) \leq f(x) < \frac{3}{2}Lg(x),$$

for all $x > a_1$. Then if $\lim_{x\to a} f(x) = \infty$, for all $M \in \mathbb{R}$, there is a $K > M$ such that if $x > K$, $f(x) > M$. Further, $g(x) > 2f(x)/3L > 2M/3L$. If $\lim_{x\to a} g(x) = \infty$, $g(x) > M \Rightarrow f(x) > Lg(x)/2 > ML/2$. $\square$